# Tensor Ranks and Matrix Multiplication Complexity

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# Matrix Multiplication Complexity

- Optimal arithmetic operations multiplying two $\mathbf{n} \times \mathbf{n}$ matrices together?
- Naively: need $O(\mathbf{n}^3)$ operations.
- Strassen's algorithm uses $O(\mathbf{n}^{2.81})$ operations.
- Matrix Multiplication Map:

$$M_{\langle l,m,n \rangle} : \mathbb{C}^{lm} \times \mathbb{C}^{mn} \to \mathbb{C}^{ln}$$
$$(A, B) \mapsto AB.$$

$$M_{\langle n \rangle} = M_{\langle n,n,n \rangle}.$$

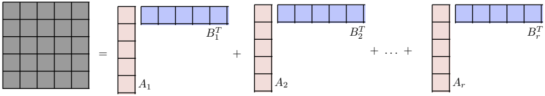
| | | |
|---|---|---|
| Multi-linear Maps | $\leftrightarrow$ | Tensors |
| Complexity of Matrix Multiplication | $\leftrightarrow$ | Tensor Rank of $M_{\langle n \rangle}$ |

- Astounding conjecture: Exponent of matrix multiplication $= 2$. Tensor rank of $M_{\langle n \rangle}$ is $O(n^{2+\epsilon})$ for any $\epsilon > 0$.

# From Linear Algebra to Multilinear Algebra

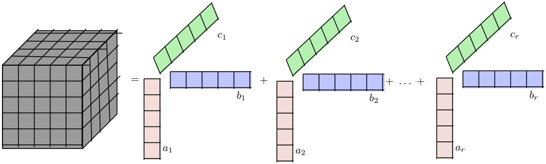
One Mode Tensor: $(T_i) \in V$.

Two Mode Tensor: $(T_{ij}) \in V_1 \otimes V_2$. Tensor Rank = Matrix Rank: Minimal $r$ s.t. $T = A_1 B_1^T + \ldots + A_r B_r^T$.




Three Mode Tensor: $(T_{ijk}) \in V_1 \otimes V_2 \otimes V_3$. Rank 1 tensor: $a \otimes b \otimes c$.

Tensor Rank: Minimal $r$ s.t. $T = a_1 \otimes b_1 \otimes c_1 + \ldots + a_r \otimes b_r \otimes c_r$.




$N$ Mode Tensor: $(T_{i_1 i_2 \ldots i_N}) \in V_1 \otimes V_2 \otimes \ldots V_N$.

Rank 1 tensor: $a_1 \otimes a_2 \otimes \ldots \otimes a_N$.

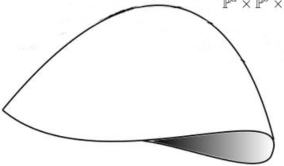
# Best Low Rank Approximation?

An ill-posed question:
A rank 3 tensor with no best rank 2 approximation:

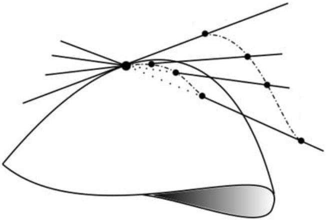
$$
\begin{aligned}
& a_1 \otimes b_1 \otimes c_2 + a_1 \otimes b_2 \otimes c_1 + a_2 \otimes b_1 \otimes c_1 \\
= \; & \lim_{t \to 0} \frac{1}{t} [(a_1 + t a_2) \otimes (b_1 + t b_2) \otimes (c_1 + t c_2) - a_1 \otimes b_1 \otimes c_1].
\end{aligned}
$$

Reason: Set of tensors of rank $\leq r$ is not closed.

# Border Ranks and Secant Varieties



Segre Embedding:
$\mathbb{P}^a \times \mathbb{P}^b \times \mathbb{P}^c \to \mathbb{P}^{(a+1)(b+1)(c+1)-1}$.



$r$-th secant variety of $X$:

$$\sigma_r(X) = \overline{\bigcup \{\langle R \rangle \colon R = \{x_1, \ldots, x_r\}, x_i \in X\}}.$$

Definition: *Border Rank*

$$\underline{\mathbf{r}}_X(F) = \min_r \{F \in \sigma_r(X)\} = \min_r \{F \in \lim_{t \to 0} \langle x_1(t), \ldots, x_r(t) \rangle\}.$$

# Rank Lower Bound Methods: Representation Theory

- $a_i, b_j, c_k$: standard basis for $A, B, C$. $\alpha_i, \beta_j, \gamma_k$: dual basis.

Equations for Secant varieties:
Two mode tensor:

$$\begin{array}{rcl} T \in A \otimes B & \leftrightarrow & L_T\colon A^* \to B \\ (T_{ij}) & \leftrightarrow & \alpha_i \mapsto \sum_i T_{ij} b_j \end{array}$$

$\mathrm{GL}(A) \times \mathrm{GL}(B) \curvearrowright (T_{ij})$. $\rightsquigarrow$ $\mathrm{GL}(A) \times \mathrm{GL}(B) \curvearrowright \sigma_r(\mathrm{Seg}(\mathbb{P}A \times \mathbb{P}B))$: space of border rank $\leq r$ tensors. $\rightsquigarrow$ $\mathrm{GL}(A) \times \mathrm{GL}(B) \curvearrowright \mathcal{I}_{\sigma_r(\mathrm{Seg}(\mathbb{P}A \times \mathbb{P}B))}$: space of polys vanishing on space of border rank $\leq r$ tensors.

- $\mathcal{I}_{\sigma_r(\mathrm{Seg}(\mathbb{P}A \times \mathbb{P}B))} = \langle (r+1) \times (r+1)$minors of $L_T \rangle = \langle \wedge^{r+1} A^* \otimes \wedge^{r+1} B^* \rangle$.
- $\wedge^{r+1} A^* \otimes \wedge^{r+1} B^* \subset \mathrm{Sym}^{r+1}(A^* \otimes B^*)$: irreducible repn.

# Why Representation Theory

- $a_i, b_j, c_k$: standard basis for $A, B, C$. $\alpha_i, \beta_j, \gamma_k$: dual basis.

Equations for Secant varieties:

Three mode tensor:

$T \in A \otimes B \otimes C \leftrightarrow T_A\colon A^* \to B \otimes C$ where $T_A(\alpha_i) = \sum_{j,k} T_{ijk} b_j \otimes c_k$.

- $\langle \wedge^{r+1} A^* \otimes \wedge^{r+1}(B \otimes C)^* \rangle \subsetneq \mathcal{I}(\sigma_r(\mathrm{Seg}(\mathbb{P}A \times \mathbb{P}B \times \mathbb{P}C)))$.
- Strassen's Equation and *Koszul Flattening*:
  Idea: $X \subset \mathbb{P}(A \otimes B \otimes C)$, $f\colon A \otimes B \otimes C \hookrightarrow U \otimes W$ linear.
  If $\forall x \in \hat{X}$, $\mathrm{rank}(f(x)) \leq t$. Then $\forall [m] \in \sigma_r(X)$,$\mathrm{rank}(f(m)) \leq rt$,
  i.e. size $rt+1$ minors of $f$ spans a subspace of $\mathcal{I}(\sigma_r(X))$.
  *Koszul flattening*: $A \hookrightarrow \mathrm{Hom}(\wedge^p A, \wedge^{p+1} A) = \wedge^p A^* \otimes \wedge^{p+1} A$. $\rightsquigarrow$
  $A \otimes B \otimes C \hookrightarrow \mathrm{Hom}(\wedge^p A \otimes B^*, \wedge^{p+1} A \otimes C)$.
  $p = 1$ Koszul flattening $\implies$ Strassen's equations.

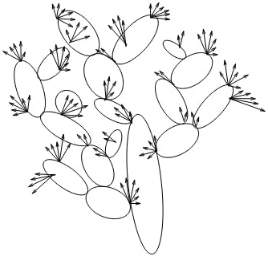
# Secant Variety and Cactus Variety

$r$-th secant variety of $X$:

$$\sigma_r(X) = \overline{\bigcup \{\langle R \rangle : R = \{x_1, \ldots, x_r\}, x_i \in X\}}.$$

$r$-th cactus variety of $X$:

$$\mathfrak{K}_r(X) = \overline{\bigcup \{\langle R \rangle : R \text{ is a (Gorenstein) subscheme of } X \text{ of length} \leq r\}}.$$



# Cactus Barrier

Three Mode Tensors:

- $\mathfrak{R}_r(\mathrm{Seg}(\mathbb{P}^n \times \mathbb{P}^n \times \mathbb{P}^n)) = \mathbb{P}((\mathbb{C}^{n+1})^{\otimes 3})$, $r = \mathcal{O}(n)$.
- $\sigma_r(\mathrm{Seg}(\mathbb{P}^n \times \mathbb{P}^n \times \mathbb{P}^n)) = \mathbb{P}((\mathbb{C}^{n+1})^{\otimes 3})$, $r = \mathcal{O}(n^2)$.

All determinantal equations before are equations of the cactus variety.
Some recent progress related to crossing cactus barrier:

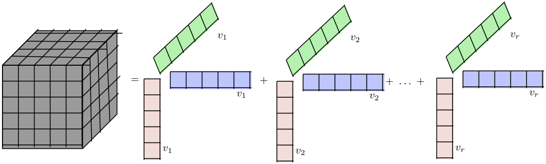
- (Gałazka-Mańdziuk-Rupniewski,2020) An algorithm is found to distinguish 14-th cactus variety and secant variety for symmetric tensors.
- (H-Michałek-Ventura,2020) A criterion is found to distinguish cactus rank, smoothable rank smoothable rank and border rank for symmetric tensors which are concise and of minimal border rank.

# Symmetric Tensors

Two Mode: Matrix $M$ is symmetric if $M = M^T$ ($M_{ij} = M_{ji}$).
Three Mode: $T$ is symmetric if its entries are the same under permuting indices:

$$T_{ijk} = T_{jik} = T_{jki} = T_{ikj} = T_{kji} = T_{kij}.$$

e.g.: partial derivatives of smooth functions
Rank 1 Symmetric Tensor: $v \otimes v \otimes v$, i.e. $T_{ijk} = v_i v_j v_k$.
Symmetric rank of $T$: minimal $r$ s.t. $T$ is a sum of rank 1 symmetric tensors.



# Symmetric Tensors and Polynomials

$$\begin{array}{rcl}
\text{symmetric tensors} & \leftrightarrow & \text{homogeneous polynomials} \\
\text{size } n \times n \times \ldots \times n \; (d \text{ times}) & \leftrightarrow & \text{degree } d \text{ in } n \text{ variables} \\
T & \leftrightarrow & \sum_{i_1, i_2, \ldots, i_d} T_{i_1, i_2, \ldots, i_d} x_{i_1} x_{i_2} \ldots x_{i_d} \\
\text{symmetric rank 1 tensor } \mathbf{v}^{\otimes d} & \leftrightarrow & \ell^d = (v_1 x_1 + \ldots + v_n x_n)^d \\
\sum_{i=1}^{r} \mathbf{v_i}^{\otimes d} & \leftrightarrow & \text{Waring rank decomposition } \sum_{i=1}^{r} \ell_i^d
\end{array}$$

# Other Notions of Ranks

- *Cactus Rank*:

$$\mathrm{cr}_X(F) = \min_r \{F \in \langle R \rangle : R \text{ is a subscheme of length } r \text{ in } X\}.$$

- *Smoothable Rank*:

$$\mathrm{sr}_X(F) = \min_r \{F \in \langle R \rangle : R \text{ is a smoothable subscheme of length } r \text{ in } X\}.$$

- *Rank*:

$$\begin{aligned} \mathrm{r}_X(F) &= \min_r \{F \in \langle R \rangle : R \text{ is smooth subscheme of length } r \text{ in } X\} \\ &= \min_r \{F \in \langle R \rangle : R = \{x_1, \ldots, x_r\}, x_i \in X\}. \end{aligned}$$

- $\mathrm{cr}_X(F) \leq \mathrm{sr}_X(F) \leq \mathrm{r}_X(F)$.

# Other Notions of Ranks

- *Cactus Rank*:

$$\mathrm{cr}_X(F) = \min_r \{F \in \langle R \rangle \colon R \text{ is a subscheme of length } r \text{ in } X\}.$$

- *Smoothable Rank*:

$$\mathrm{sr}_X(F) = \min_r \{F \in \langle R \rangle \colon R \text{ is a smoothable subscheme of length } r \text{ in } X$$
$$= \min_r \{F \in \langle \lim_{t\to 0} R(t) \rangle = \langle \lim_{t\to 0} x_1(t), \ldots, x_r(t) \rangle\}.$$

- *Border Rank*:

$$\underline{\mathbf{r}}_X(F) = \min_r \{F \in \sigma_r(X)\} = \min_r \{F \in \lim_{t\to 0} \langle x_1(t), \ldots, x_r(t) \rangle\}.$$

- $\mathrm{cr}_X(F) \leq \mathrm{sr}_X(F) \leq \mathrm{r}_X(F)$. $\mathrm{cr}_X(F) \leq \mathrm{sr}_X(F) \leq \mathrm{r}_X(F)$.
- $\underline{\mathbf{r}}_X(F) \leq \mathrm{sr}_X(F) \leq \mathrm{r}_X(F)$. ($\lim_{t\to 0}\langle R(t) \rangle \supset \langle \lim_{t\to 0} R(t) \rangle$)
  $\underline{\mathbf{r}}_X(F) \leq \mathrm{sr}_X(F) \leq \mathrm{r}_X(F)$. ($\lim_{t\to 0}\langle R(t) \rangle \supset \langle \lim_{t\to 0} R(t) \rangle$)

# Cactus Rank and Border Rank

- $\underline{\mathbf{r}}_X(F) \leq \mathrm{sr}_X(F) \leq \mathrm{r}_X(F)$.
- $\mathrm{cr}_X(F) \leq \mathrm{sr}_X(F) \leq \mathrm{r}_X(F)$.
- $\mathrm{cr}_X(F)$ and $\underline{\mathbf{r}}_X(F)$ are not comparable :
  - $\mathrm{cr}_X(F) < \underline{\mathbf{r}}_X(F)$: $X \subset \mathbb{P}^N$ is a curve with a singularity $p \in X$ such that $T_pX = \mathbb{P}^N$.
    For a generic $F \in \mathbb{P}^N$, $\mathrm{cr}_X(F) = 2$ while $\underline{\mathbf{r}}_X(F) \leq \mathrm{sr}_X(F)$ could be arbitrarily large if $N \gg 0$.
  - $\mathrm{cr}_X(F) = \mathrm{sr}_X(F) > \underline{\mathbf{r}}_X(F)$: $X = \mathbb{P}A \times \mathbb{P}B \times \mathbb{P}C \subset \mathbb{P}(A \otimes B \otimes C)$.
    $F = a_2 \otimes b_1 \otimes c_2 + a_2 \otimes b_2 \otimes c_1 + a_1 \otimes b_1 \otimes c_3 + a_1 \otimes b_3 \otimes c_1 + a_3 \otimes b_1 \otimes c_1$.
    $\underline{\mathbf{r}}_X(F) = 3$ while $\mathrm{sr}_X(F) = \mathrm{cr}_X(F) = 4$.
- Higher border rank examples when $\underline{\mathbf{r}}_X(F) < \mathrm{sr}_X(F)$?

# Polynomials: Tame and Wild

- $X = \nu_d(\mathbb{P}V) \subset \mathbb{P}^{N_d}$, $V \cong \mathbb{C}^{n+1}$ and $F \in S^d V$.

**Definition**

$F$ is *wild* if $\mathrm{sr}_X(F) > \underline{\mathbf{r}}_X(F)$. Otherwise, we say $F$ is *tame*.

- Classical: $F$ is tame if $n = 1$.
- (Buczyńska, Buczyński) If $\underline{\mathbf{r}}(F) \leq \max\{4, d+1\}$, then $F$ is tame.

**Theorem (Buczyńska, Buczyński,2014)**

*For cubic polynomials, $F$ is tame if $n \leq 3$.*

# First Example of a Wild Cubic

Example(Buczyńska, Buczyński,2014)

$$F = x_0 x_1^2 - x_2(x_1 + x_4)^2 + x_3 x_4^2.$$

$F$ is a wild cubic with $\underline{\mathbf{r}}(F) = 5$ and $\mathrm{cr}(F) = \mathrm{sr}(F) = 6$.

$$F = \lim_{t \to 0} (\frac{1}{3}(x_1 + tx_0)^3 - \frac{1}{3}((x_1 + x_4) + tx_2)^3 + \frac{1}{12}(2x_4 - tx_2)^3 - \frac{1}{9}(x_1 - x_4)^3 + \frac{1}{9}(x_1 + 2x_4)^3).$$

# Polynomials of Vanishing Hessian

## Definition

$F \in S^d V$ is *a polynomial with vanishing Hessian* if
$\mathrm{Hess}(F) = \det([\frac{\partial F}{\partial x_i \partial x_j}]) = 0$.

## Easy Fact

$F$ is *a polynomial with vanishing Hessian* if and only if $\{\frac{\partial F}{\partial x_0}, \dots, \frac{\partial F}{\partial x_n}\}$ are algebraically dependent.

- Wild Example: $F = x_0 x_1^2 - x_2(x_1 + x_4)^2 + x_3 x_4^2$.

## Question(Ottaviani)

$F$ is a concise polynomial with vanishing Hessian.
Is there a relation between wild polynomials and polynomials with vanishing Hessian?

# Vanishing Hessian and Wild Polynomials

- *concise*: $F \in S^d V$ is *concise* if $F\colon V^* \to S^{d-1}V$ is of full rank.
- *minimal border rank*: $F$ is *of minimal border rank* if $\underline{\mathbf{r}}(F) = \#$ of variables $=$ dim of each mode.

**Theorem (H-Michałek-Ventura,2020)**

*Let $d \geq 3$ and $F \in S^d V$ be a concise polynomial of minimal border rank. Then:*

$$\mathrm{Hess}(F) = 0 \iff \mathrm{cr}(F) > \underline{\mathbf{r}}(F) \iff \mathrm{sr}(F) > \underline{\mathbf{r}}(F).$$

- First time explicit equations are found to distinguish $\mathrm{sr}, \mathrm{cr}$ and $\underline{\mathbf{r}}$.

# Outline of Part II

1. New technique: Border Apolarity and its application
2. History of estimates of matrix multiplication complexity
3. Border rank of $3 \times 3$ permanent: Why we care and how to compute it
4. Recent and future directions

# Break?

# Border Apolarity(BB,2019): Tensor Case

- $T \in A \otimes B \otimes C$,
  $S = \mathrm{Sym}(A \oplus B \oplus C)^* = \bigoplus_{s,t,u} S^s A^* \otimes S^t B^* \otimes S^u C^*$ is $\mathbb{Z}^3$-graded coordinate ring of $\mathrm{Seg}(\mathbb{P}A \times \mathbb{P}B \times \mathbb{P}C)$.
- $\mathrm{Ann}(T) := \{\Theta \in S \mid \Theta \lrcorner T = 0\}$ is a $\mathbb{Z}^3$-graded ideal of $S$.
- Apolarity: $r(T) \leq r \iff \exists$ multi-graded ideal $\mathcal{I} \subset \mathrm{Ann}(T)$ such that $\mathcal{I}$ defines $r$ distinct points in $\mathrm{Seg}(\mathbb{P}A \times \mathbb{P}B \times \mathbb{P}C)$.
- Border apolarity: $\underline{\mathbf{r}}(T) \leq r \iff \exists$ multi-graded ideal $\mathcal{I} \subset \mathrm{Ann}(T)$ with Hilbert function $h_{S/\mathcal{I}}(s,t,u) = \min\{r, \dim S^s A^* \otimes S^t B^* \otimes S^u C^*\}$ such that $\mathcal{I}$ is a limit of radical ideals of $r$ distinct points in $\mathrm{Seg}(\mathbb{P}A \times \mathbb{P}B \times \mathbb{P}C)$.
- Hilbert function: $h_{S/\mathcal{I}}(s,t,u) := \dim(S^s A^* \otimes S^t B^* \otimes S^u C^* / \mathcal{I}_{s,t,u})$.

# Border Apolarity(cont'd)

- Idea: border rank decomposition of $T = \lim_{t\to 0}\sum_{j=1}^{r} T_j(t) \rightsquigarrow \mathcal{I}_t =$ ideal of $[T_1(t)] \sqcup \ldots \sqcup [T_r(t)]$ is a radical ideals of $r$ distince points in $\mathrm{Seg}(\mathbb{P}A \times \mathbb{P}B \times \mathbb{P}C)$ when $t \neq 0 \rightsquigarrow$ take $t \to 0$ get $\mathcal{I}_0 = \lim_{t\to 0}\mathcal{I}_t$ is the desired ideal.
- Example: $T \in \mathbb{C}^2 \otimes \mathbb{C}^2 \otimes \mathbb{C}^2$.
  $T = a_0 \otimes b_0 \otimes c_1 + a_0 \otimes b_1 \otimes c_0 + a_1 \otimes b_0 \otimes c_0 =$
  $\lim_{t\to 0}\frac{1}{t}[(a_0 + ta_1) \otimes (b_0 + tb_1) \otimes (c_0 + tc_1) - a_0 \otimes b_0 \otimes c_0)]$.
  $\mathcal{I}_0 = \lim_{t\to 0}\mathcal{I}_t((a_0 + ta_1) \otimes (b_0 + tb_1) \otimes (c_0 + tc_1), a_0 \otimes b_0 \otimes c_0) =$
  $\langle \alpha_1^2, \beta_1^2, \gamma_1^2, \alpha_1\beta_1, \alpha_1\gamma_1, \beta_1\gamma_1, \alpha_1\beta_0 - \alpha_0\beta_1, \alpha_1\gamma_0 - \alpha_0\gamma_1, \beta_0\gamma_1 - \beta_1\gamma_0\rangle$.

# Why Constant Hilbert Function

- How to ensure $h_{S/\mathcal{I}}(s,t,u) = \min\{r, \dim S^s A^* \otimes S^r B^* \otimes S^u C^*\}$?
- Border rank decomposition can be flexible:

$$\begin{aligned} & a_1 \otimes b_1 \otimes c_2 + a_1 \otimes b_2 \otimes c_1 + a_2 \otimes b_1 \otimes c_1 \\ = \ & \lim_{t\to 0} \frac{1}{t}[(a_1 + ta_2) \otimes (b_1 + tb_2) \otimes (c_1 + tc_2) - a_1 \otimes b_1 \otimes c_1] \\ = \ & \lim_{t\to 0} \frac{1}{t}[(a_1 + ta_2 + t^2 a_3 + \dots) \otimes (b_1 + tb_2 + t^2 b_3 + \dots) \\ & \otimes (c_1 + tc_2 + t^2 c_3 + \dots) - a_1 \otimes b_1 \otimes c_1] \quad . \end{aligned}$$

- WLOG, can assume points are in general position when $t > 0 \rightsquigarrow$ Hilbert function of $S/\mathcal{I}_t$ is almost constant.

# What Deformation Theory Could Do

Summary: If $\underline{\mathbf{r}}(T) \leq r$, we can find ideals satisfying
(1) $\mathcal{I} \subset \mathrm{Ann}(T)$
(2) $h_{S/\mathcal{I}}(s,t,u) = \min\{r, \dim S^s A^* \otimes S^r B^* \otimes S^u C^*\}$.
Bonus: If $T$ has symmetry, can ask for a Borel fixed $\mathcal{I}$.
$\rightsquigarrow$ all programmable polynomial conditions. $\rightsquigarrow$ candidate ideals satisfying (1) and (2).
(3) $\mathcal{I}$ is a limit of radical ideals of $r$ distinct points : Need finer information about multi-graded Hilbert scheme.

# Back to Matrix Multiplication Complexity

- Previously, only known $\underline{\mathbf{r}}$ for nontrivial $M_{\langle l,m,n\rangle}$ is
$$\underline{\mathbf{r}}(M_{\langle 2\rangle}) = 7.$$
- Using border apolarity (Conner-Harper-Landsberg,2019):
$$\underline{\mathbf{r}}(M_{\langle 2,2,3\rangle}) = 10$$
$$\underline{\mathbf{r}}(M_{\langle 2,3,3\rangle}) = 14.$$
- Previously known: $\underline{\mathbf{r}}(M_{\langle 2,n,n\rangle}) \geq n^2 + 1$ (Lickteig).
Using border apolarity: $\underline{\mathbf{r}}(M_{\langle 2,n,n\rangle}) \geq n^2 + 1.32n + 1$ for $n > 25$.
- Previously known: $\underline{\mathbf{r}}(M_{\langle 3,n,n\rangle}) \geq n^2 + 2$ (Lickteig).
Using border apolarity: $\underline{\mathbf{r}}(M_{\langle 3,n,n\rangle}) \geq n^2 + 2n$ for $n > 14$.
- Exponent of matrix multiplication:
$\omega := \inf_\tau\{\mathbf{n} \times \mathbf{n}$ matrices can be multiplied using $O(\mathbf{n}^\tau)$ arithmetic operations $\}$.
$= \inf_\tau\{\underline{\mathbf{R}}(M_{\langle n\rangle}) = O(\mathbf{n}^\tau)\}$ (Strassen, Bini).
Astounding conjecture: $\omega = 2$.

# History of Estimates of $\omega$

| Upper bound of $\omega$ | Year | Authors |
|---|---|---|
| 3 | | |
| 2.81 | 1969 | Strassen |
| 2.79 | 1979 | Pan |
| 2.78 | 1979 | Bini,Capovani,Romani and Lotti |
| 2.55 | 1981 | Schöhage: Asymptotic Sum Inequality |
| 2.53 | 1981 | Pan |
| 2.52 | 1982 | Romani |
| 2.50 | 1982 | Coppersmith and Winograd |
| 2.48 | 1986 | Strassen: Laser Method |
| 2.376 | 1987 | Coppersmith and Winograd |
| 2.373 | 2010 | Stothers |
| 2.3729 | 2012 | Vassilevska Williams |
| 2.37287 | 2014 | Le Gall |
| 2.37286 | 2020 | Alman-Williams |

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# Schönhage's Asymptotic Sum Inequality

- Idea: If $\underline{\mathbf{r}}(M_{\langle n\rangle}^{\oplus s}) < s \times \underline{\mathbf{r}}(M_{\langle n\rangle})$ with $s \leq n^3$. To compute $M_{\langle n^2\rangle} = M_{\langle n\rangle}^{\oplus n^3}$, can group each $s$ copies to obtain savings.
- Schöchage: sometimes

$$\underline{\mathbf{r}}(M_{\langle l,m,n\rangle} \oplus M_{\langle l',m',n'\rangle}) \ll \underline{\mathbf{r}}(M_{\langle l,m,n\rangle}) + \underline{\mathbf{r}}(M_{\langle l',m',n'\rangle})$$

.

$T_i \in A_i \otimes B_i \otimes C_i$, define *Kronecker product*
$T_1 \boxtimes T_2 \in (A_1 \otimes A_2) \otimes (B_1 \otimes B_2) \otimes (C_1 \otimes C_2)$ and *Kronecker powers*
$T^{\boxtimes k} \in (A^{\otimes k}) \otimes (B^{\otimes k}) \otimes (C^{\otimes k})$.
$(M_{\langle l,m,n\rangle} \oplus M_{\langle l',m',n'\rangle})^{\boxtimes 2} = M_{\langle l^2,m^2,n^2\rangle} \oplus M_{\langle ll',mm',nn'\rangle}^{\oplus 2} \oplus M_{\langle l'^2,m'^2,n'^2\rangle}$.
Fact: $M_{\langle l,m,n\rangle} \boxtimes M_{\langle l',m',n'\rangle} = M_{\langle ll',mm',nn'\rangle}$. $\rightsquigarrow \omega < 2.55$.

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| 2.52 | 1982 | Romani |
| 2.50 | 1982 | Coppersmith and Winograd |
| 2.48 | 1986 | Strassen: Laser Method |
| 2.376 | 1987 | Coppersmith and Winograd |
| 2.373 | 2010 | Stothers |
| 2.3729 | 2012 | Vassilevska Williams |
| 2.37287 | 2014 | Le Gall |
| 2.37286 | 2020 | Alman-Williams |

# Strassen's Laser Method

- Laser Method: fix a "good" auxiliary tensor $T$ and degenerate its high tensor powers $T^{\boxtimes N}$ to a disjoint union of large matrix multiplication tensors. Good bound on $\underline{\mathbf{r}}(T^{\boxtimes N}) \quad \rightsquigarrow$ good bound of $\omega$.
- Current World Record(Stouthers,William,LeGall,Alman-Williams,2020): show $\omega < 2.37286$ using big Coppersmith-Winograd tensor:

$$\begin{aligned} T_{CW,q} &= a_0 \otimes b_0 \otimes c_{q+1} + a_0 \otimes b_{q+1} \otimes c_0 + a_{q+1} \otimes b_0 \otimes c_0 \\ &\quad + \sum_{j=1}^{q} (a_0 \otimes b_j \otimes c_j + a_j \otimes b_0 \otimes c_j + a_j \otimes b_j \otimes c_0) \\ &\in (\mathbb{C}^{q+2})^{\otimes 3}. \end{aligned}$$

- Existing Barrier: can never use $T_{CW,q}$ to show $\omega < 2.37$.
- $\underline{\mathbf{r}}(T_{CW,q}) = q + 2$. No tensors of minimal border rank can be used in Laser method to show $\omega = 2$.

# Small Coppersmith-Winograd Tensor

- Good tensors that do not subject to this barrier using Laser method. One of them: small Coppersmith-Winograd tensor:

$$T_{cw,q} = \sum_{j=1}^{q} (a_0 \otimes b_j \otimes c_j + a_j \otimes b_0 \otimes c_j + a_j \otimes b_j \otimes c_0) \in (\mathbb{C}^{q+1})^{\otimes 3}.$$

  for $2 \leq q \leq 10$.
  $q = 2$: only known tensor that could potentially prove $\omega = 2$.

- Upper bounds of $\omega$ using $T_{cw,q}$ :

$$\omega \leq \log_q\left(\frac{4}{27}(\underline{\mathbf{r}}(T_{cw,q}^{\boxtimes k}))^{\frac{3}{k}}\right).$$

- Known: $\underline{\mathbf{r}}(T_{cw,q}) = q + 2$. $k = 1, q = 8$ gives $\omega < 2.41$.
- Get better bound if $\underline{\mathbf{r}}(T_{cw,q}^{\boxtimes k}) < (\underline{\mathbf{r}}(T_{cw,q}))^k$.

# Small Coppersmith-Winograd Tensor: Knows and Unknowns

(Conner,Gesmundo,Landsberg,Ventura,2019):

- $\underline{\mathbf{r}}(T_{cw,q}^{\boxtimes 2}) = (q+2)^2$ for $q > 2$ and $15 \leq \underline{\mathbf{r}}(T_{cw,2}^{\boxtimes 2}) \leq 16$.
- $\underline{\mathbf{r}}(T_{cw,q}^{\boxtimes 3}) = (q+2)^3$ for $q > 4$.
- For all $q > 4$ and all $k$, $\underline{\mathbf{r}}(T_{cw,q}^{\boxtimes k}) \geq (q+2)^3(q+1)^{k-3}$ and $\underline{\mathbf{r}}(T_{cw,4}^{\boxtimes k}) \geq 36(5)^{k-2}$.

Note: $T_{cw,2}^{\boxtimes 2} = \mathrm{perm}_3$, the $3 \times 3$ permanent considered as a tensor.

**Theorem (Conner-Landsberg-H,2020)**

$\underline{\mathbf{r}}(T_{cw,2}^{\boxtimes 2}) = 16 = (\underline{\mathbf{r}}(T_{cw,2}))^2$.

# Border Apolarity and Flag Condition

- Challenge to apply Border Apolarity when $T = \mathrm{perm}_3$: Borel subgroup in stablizer of $[T]$ is $\mathbb{T}^6$: 6-dim'l torus. Need to introduce numerous parameters when searching candidate limiting ideals even in degree (110). $\rightsquigarrow$ infeasible calculation.

**Flag Condition**

If $\mathcal{I}$ is a limiting ideal from a border rank $r$ decomposition, then there exists a $\mathbb{B}_T$-fixed filtration of $\mathcal{I}_{110}^{\perp}$, $F_1 \subset F_2 \subset \ldots \subset F_r = \mathcal{I}_{110}^{\perp}$, such that $F_j \subset \sigma_j(\mathrm{Seg}(\mathbb{P}A \times \mathbb{P}B))$.

- This is a generalization of the known flag condition in the concise tensor of minimal border rank case.
- The flag condition guaranteed the presence of low rank element in $\mathcal{I}_{110}^{\perp}$ which significantly reduced the search space (hand checkable).

# Towards Future

- Skew Cousin of small Coppersmith-Winograd tensor ($q = 2p$): $\color{blue}{T_{skewcw,q}} = \sum_{\xi=1}^{p} a_0 \otimes b_\xi \otimes c_{\xi+p} - a_0 \otimes b_{\xi+p} \otimes c_\xi - a_\xi \otimes b_0 \otimes c_{\xi+p} + a_{\xi+p} \otimes b_0 \otimes c_\xi + a_\xi \otimes b_{\xi+p} \otimes c_0 - a_{\xi+p} \otimes b_\xi \otimes c_0 \in \mathbb{C}^{q+1} \otimes \mathbb{C}^{q+1} \otimes \mathbb{C}^{q+1}$.
- $\omega \leq \log_q\left(\frac{4}{27}(\underline{\mathbf{r}}(T_{skewcw,q})^{\boxtimes k})^{\frac{3}{k}}\right)$.
- $\underline{\mathbf{r}}(T_{cw,q}) = q + 2$ and $\underline{\mathbf{r}}(T_{skewcw,q}) = \frac{3}{2}q + 2$.
- $\underline{\mathbf{r}}(T_{cw,q}^{\boxtimes 2}) = (\underline{\mathbf{r}}(T_{cw,q}))^2$.

| $q$ | $\underline{\mathbf{r}}(T_{skewcw,q})$ | $\underline{\mathbf{r}}(T_{skewcw,q})^2$ | $\underline{\mathbf{r}}(T_{skewcw,q}^{\boxtimes 2})$ |
|---|---|---|---|
| 2 | 5 | 25 | 17 |
| 4 | 8 | 64 | $\leq 42^*$ |
| $6 - 10$ | $\frac{3}{2}q + 2$ | $(\frac{3}{2}q + 2)^2$ | ? |

# Recent Progress using Border Apolarity

On set-theoretical defining equations of tensors of minimal border rank:

**Definition**

A tensor $T \in \mathbb{C}^m \otimes \mathbb{C}^m \otimes \mathbb{C}^m$ is of *minimal border rank* if $\underline{\mathbf{r}}(T) = m$.

Previously Known:
(Friedland, Gross, 2011) Set-theoretical version of a *Salmon conjecture*: Set-theoretical defining equations of tensors of minimal border rank when $m = 4$.
New progress:
(Jelisiejew-Landsberg-Pal) Set-theoretical defining equations of concise tensors of minimal border rank when $m = 5$ and $1_*$-generic tensors of minimal border rank when $m = 6$.

**Definition**

A tensor $T \in \mathbb{C}^m \otimes \mathbb{C}^m \otimes \mathbb{C}^m$ is *$1_*$-generic* if $T(\mathbb{C}^m) \subset \mathbb{C}^m \otimes \mathbb{C}^m$ contains an element of maximal rank.