Spiraling Domains in Dimension 2

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(Joint work in progress with Xavier Buff)

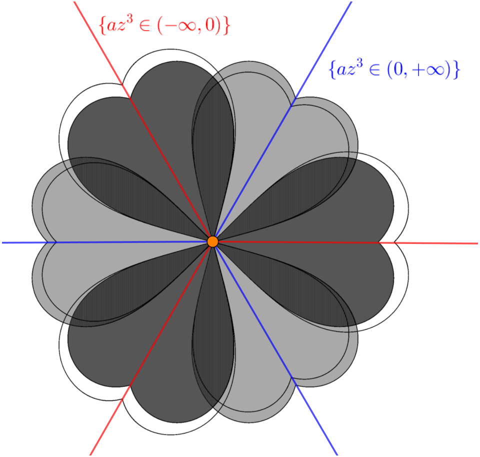
Theorem (Buff-R., in progress)

There exist $f\colon \mathbb{C}^2 \to \mathbb{C}^2$ polynomial maps tangent to the identity at the origin with infinitely many parabolic domains of spiraling type.

Dimension 1

- $f : (\mathbb{C}, 0) \to (\mathbb{C}, 0)$ tangent to the identity and $f \neq \mathrm{id}$:

$$f(z) = z + az^{k+1} + \mathcal{O}(z^{k+2}) \quad \text{with} \quad a \in \mathbb{C} \smallsetminus \{0\}.$$



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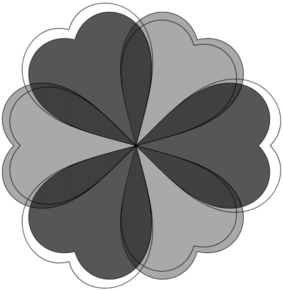

$k = 3$

f is topologically conjugate to the time-1 flow of $z^{k+1} \frac{\partial}{\partial z}$.

Dimension 2

Setting:

- $\mathbf{z} = \begin{pmatrix} x \\ y \end{pmatrix} \in \mathbb{C}^2$
- $f(\mathbf{z}) = \mathbf{z} + v(\mathbf{z}) + \mathcal{O}\left(\|\mathbf{z}\|^{k+2}\right)$, $k \geq 1$
- $v : \mathbb{C}^2 \to \mathbb{C}^2$ homogeneous map of degree $k + 1$.

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Idea: Look at $\vec{v}(\mathbf{z})$

- search for preferred directions for the dynamics
- understand orbits of f using real time trajectories of $\vec{v}(\mathbf{z})$.

$$f(\mathbf{z}) = \mathbf{z} + v(\mathbf{z}) + \mathcal{O}\left(\|\mathbf{z}\|^{k+2}\right), \quad f^{\circ n}(\mathbf{z}) = \mathbf{z}_n$$

Fact: $\mathbf{z}_n \to \mathbf{0}$ tangentially to $[\mathbf{t}] \in \mathbb{P}^1(\mathbb{C}) \Longrightarrow \exists \lambda \in \mathbb{C}$ s.t. $v(\mathbf{t}) = \lambda \mathbf{t}$.

- $[\mathbf{t}] \in \mathbb{P}^1(\mathbb{C})$ is a characteristic direction if $v(\mathbf{t}) = \lambda \mathbf{t}$. $[\mathbf{t}]$ is non-degenerate if $\lambda \neq 0$, degenerate if $\lambda = 0$.
- v is dicritical if all directions are characteristic, non-dicritical otherwise.

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From now on: v non-dicritical

- Existence of parabolic curves tangent to characteristic directions. [Écalle, Hakim, Abate, ..., López-Rosas]
- Existence of finitely many parabolic domains tangent to characteristic directions. [Écalle, Hakim, Vivas, Rong, ...]
- Existence of a spiraling domain where orbits converge to the origin not being tangent to any direction. [Rivi, Rong]

Theorem (Buff-R., in progress)

There exist $f : \mathbb{C}^2 \to \mathbb{C}^2$ polynomial maps tangent to the identity at the origin with infinitely many parabolic domains of spiraling type.

Idea: study real time trajectories of $\vec{v}(\mathbf{z})$ inside its complex time trajectories.

Theorem (Buff-R., in progress)

1. $f\begin{pmatrix}x\\y\end{pmatrix} = \begin{pmatrix}x\\y\end{pmatrix} + \begin{pmatrix}y^2\\x^2\end{pmatrix} + a(x-y)\begin{pmatrix}x\\y\end{pmatrix}$ *with* $a \in \mathbb{R} \setminus \{0\}$
2. $f\begin{pmatrix}x\\y\end{pmatrix} = \begin{pmatrix}x\\y\end{pmatrix} + \begin{pmatrix}y^2\\x^2\end{pmatrix} + \lambda xy\begin{pmatrix}x\\y\end{pmatrix}$ *with* $\lambda \in (1,\infty)$

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Orbits shadow real time trajectories of $\vec{v} = y^2\frac{\partial}{\partial x} + x^2\frac{\partial}{\partial y}$.

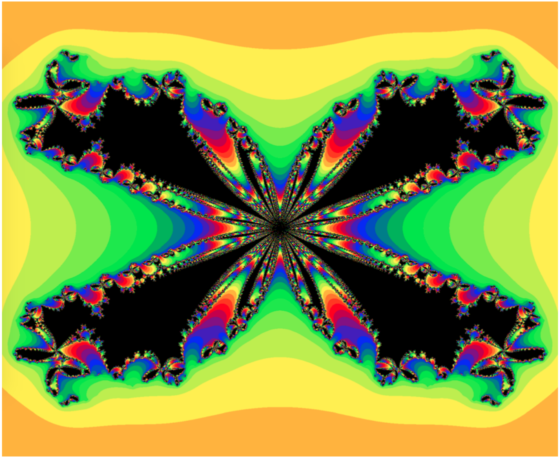
Trajectories for $\vec{v} = y^2 \frac{\partial}{\partial x} + x^2 \frac{\partial}{\partial y}$

- $\vec{v}$ is a Hamiltonian vector field
- Complex trajectories of $\vec{v}$:

$$\mathscr{S}_\kappa := \left\{ \mathbf{z} \in \mathbb{C}^2 \mid \Phi(\mathbf{z}) := x^3 - y^3 = \kappa \right\} \text{ with } \kappa \in \mathbb{C}.$$

- $\mathscr{S}_0 = \{y = x\} \cup \{y = jx\} \cup \{y = j^2 x\}$ with $j = e^{\frac{2\pi i}{3}}$
- $\mathbf{0} \notin \overline{\mathscr{S}_\kappa}$ for $\kappa \neq 0$, and so real trajectories of $\vec{v}$ in $\mathscr{S}_\kappa$ do not converge to $\mathbf{0}$.
- For $\kappa \neq 0$, $\mathscr{S}_\kappa \simeq \text{Torus} \setminus \{3 \text{ points}\}$, on which $\vec{v}$ is a translation vector field.
- If $\kappa = (p + jq)^3 r$, with $(p, q) \in \mathbb{Z}^2 \setminus \{\mathbf{0}\}$ and $r \in \mathbb{R} \setminus \{0\}$, then the real trajectories of $\vec{v}$ are periodic, that is closed.

Spiraling domains for $f\begin{pmatrix}x\\y\end{pmatrix} = \begin{pmatrix}x\\y\end{pmatrix} + \begin{pmatrix}y^2\\x^2\end{pmatrix} + 2xy\begin{pmatrix}x\\y\end{pmatrix}$



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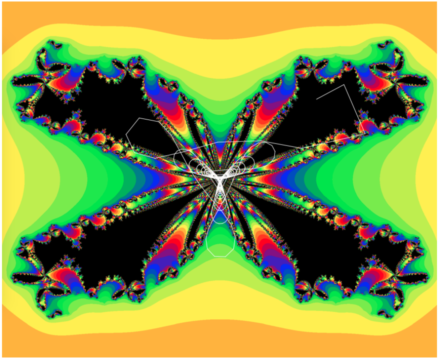

orbit

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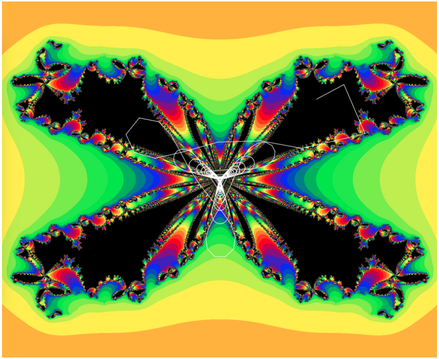

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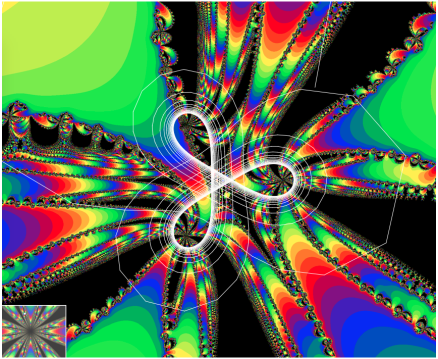

radial projection

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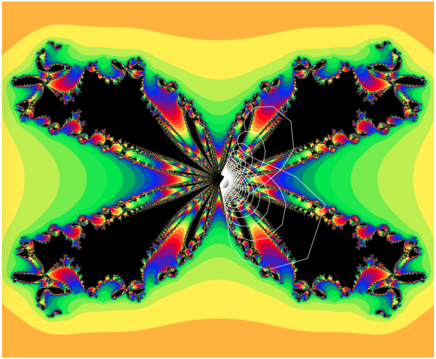

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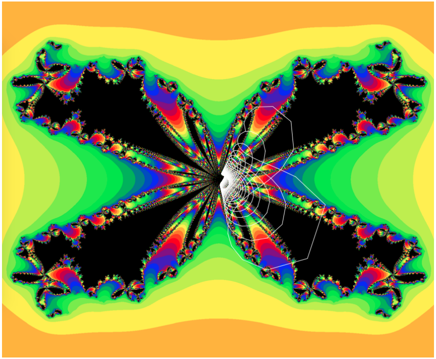

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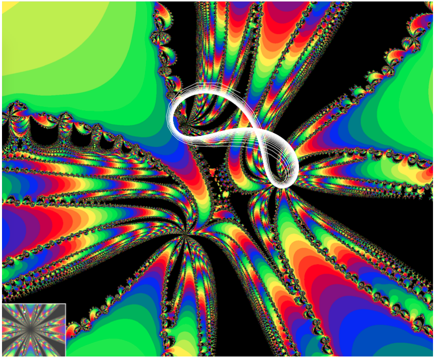

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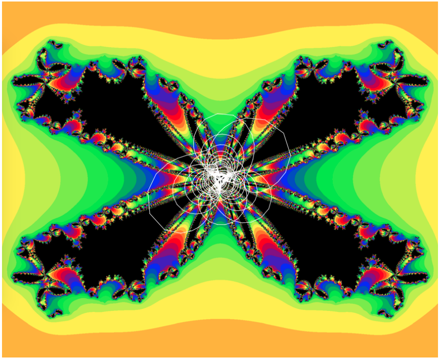

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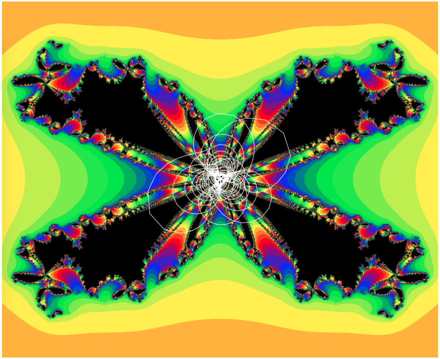

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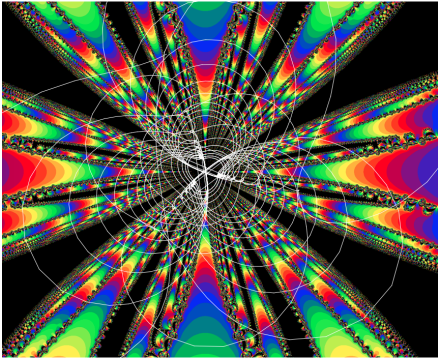

zoom

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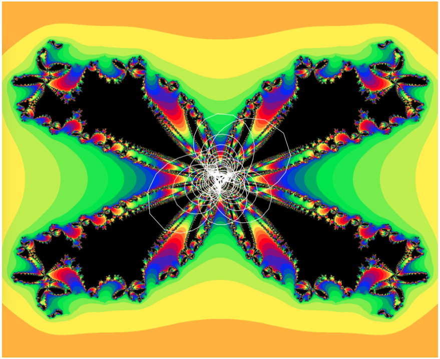

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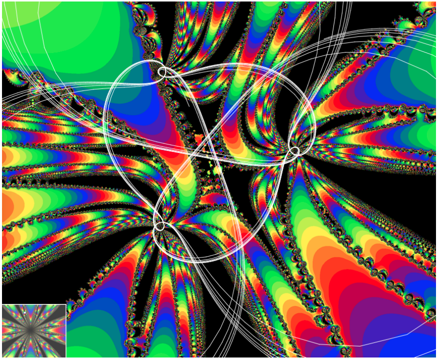

radial projection zoom

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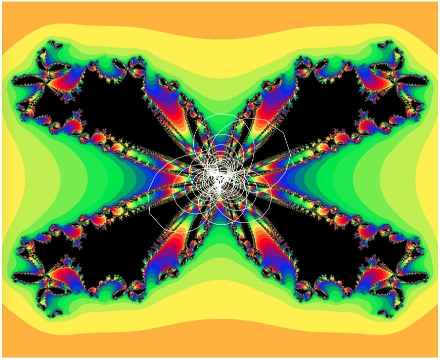

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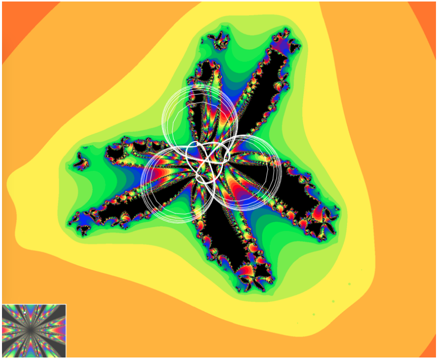

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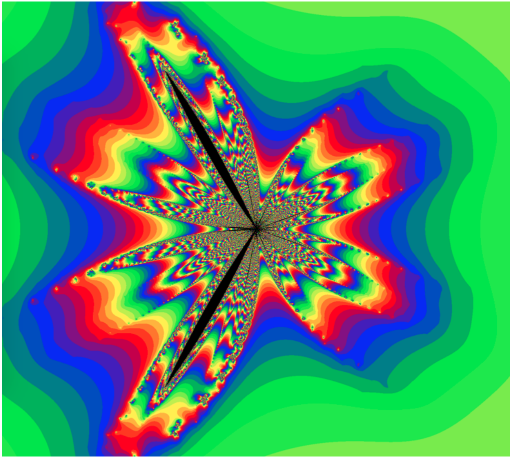

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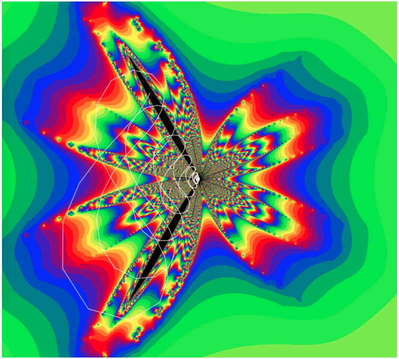

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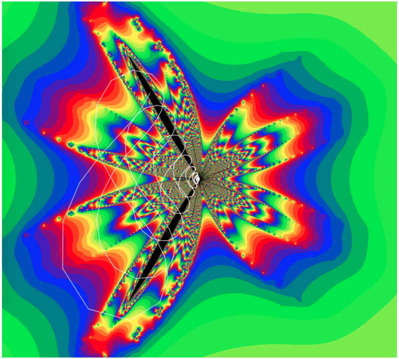

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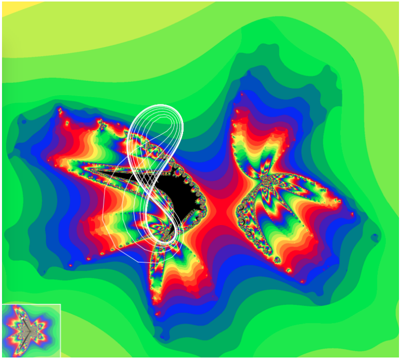

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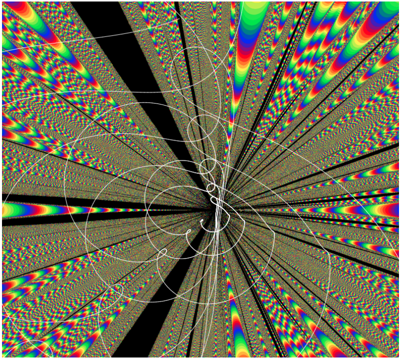

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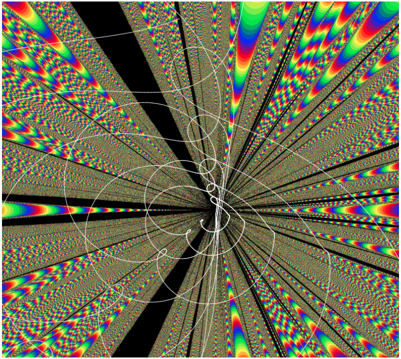

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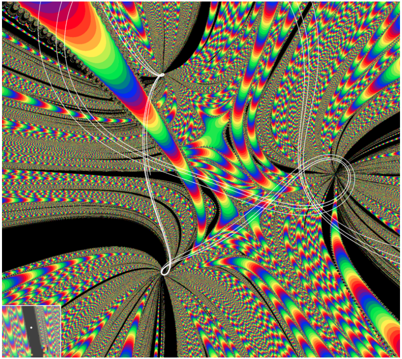

radial projection zoom

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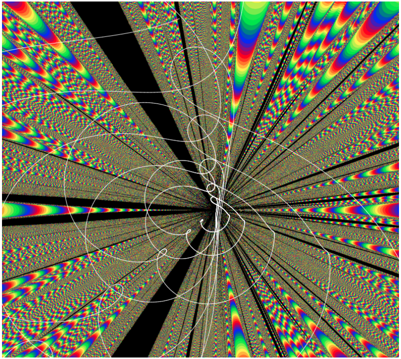

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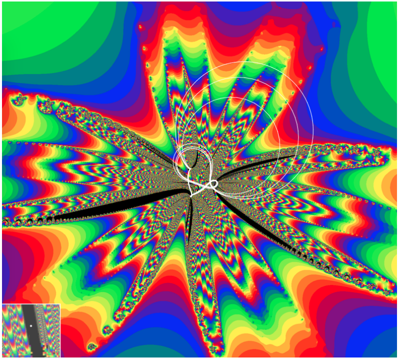

radial projection

Thanks for your attention!