I] Motivation and Program

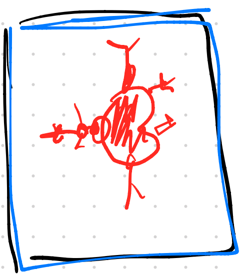
$\mathrm{Per}_n(0) = \{$ Quadratic Rational Maps with a critical point in an n-cycle $\}$

$\cup$

$\mathcal{M}_n := \{$ non-escaping parameters $\}$

Many Per_n's: in general, just a "dynamical subvariety" of some moduli space, eg $\mathrm{Poly}_3/\sim$, $\mathrm{Rat}_2/\sim$ etc

eg]



$\mathrm{Per}_1(0) := \{$ QRM w/ a superattracting f.p. $\}$

$\simeq \mathbb{C} = \{z^2+c \mid c \in \mathbb{C}\}$

$\mathcal{M}_1$ the Mandelbrot set

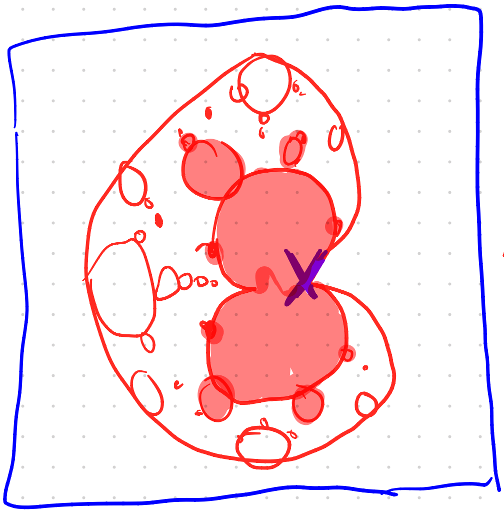
$\| $

M

$\{c \in \mathbb{C} \mid 0 \not\to \infty\}$

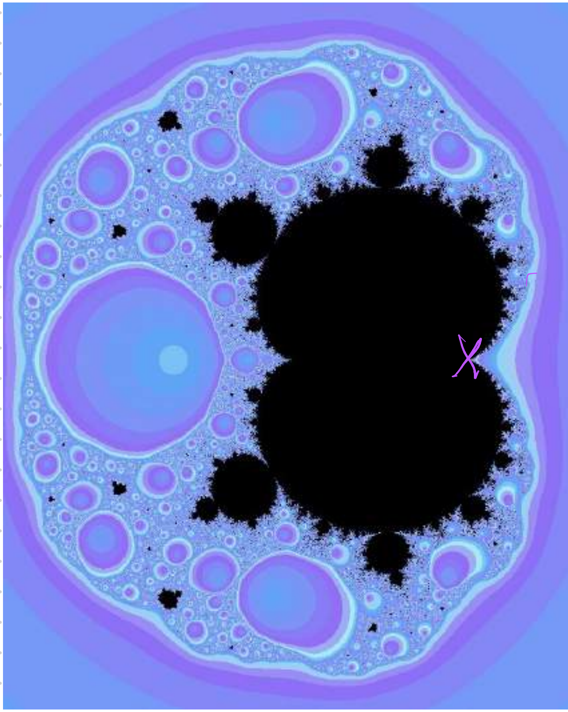
$\mathrm{Per}_2(0) := \{$ maps w/ a superattracting 2-cycle $\}$

Say: $\infty \overset{2}{\rightleftarrows} 1 \quad 0 \overset{?}{\to} \cdots$



$\mathrm{Per}_2(0) \simeq \hat{\mathbb{C}} - \{pt\}$

$$M_2 := \left\{ c \text{ st } \begin{matrix} 0 \not\to \infty \\ \& \\ 0 \not\to 1 \end{matrix} \right\}$$

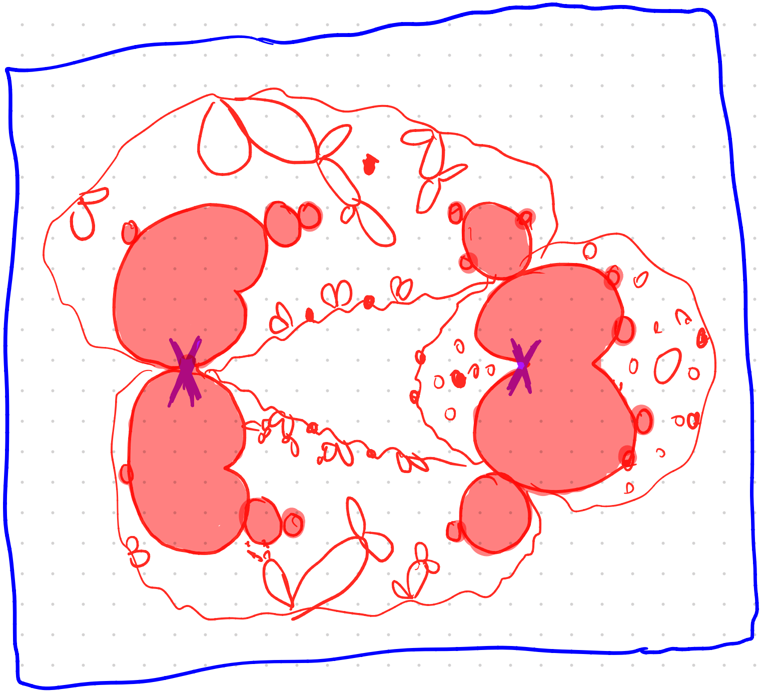


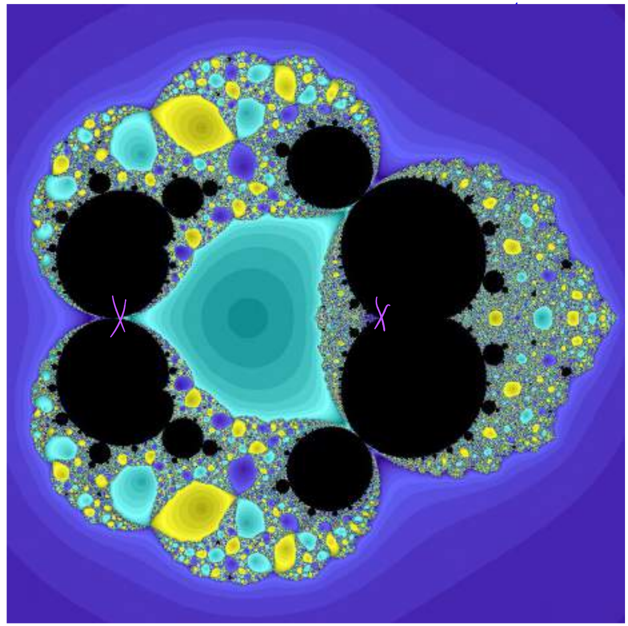
⚠ There's a blob in the middle that kind of looks like M! ⚠

Same game for higher n...

$M_3 \subset \mathrm{Per}_3 \cong \mathbb{C} - \{pt\}$ $\quad \infty \xrightarrow{3} 1 \to x$

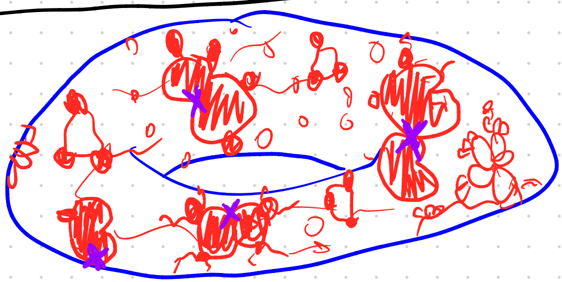
$0 \xrightarrow{3} \cdots$



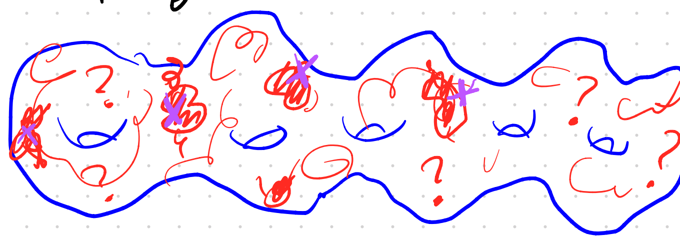


Per_5

10 X



Per_6 18 X



Questions:

① What global topology does Per_n have?

Ramadas: yes! $n \leq 19$

Epstein: Per_n smooth

↳ is it connected?!

↳ what's its genus?

↳ How many punctures does it have?

② How to organize & model $\mathcal{M}_n$?

↳ We understand well:

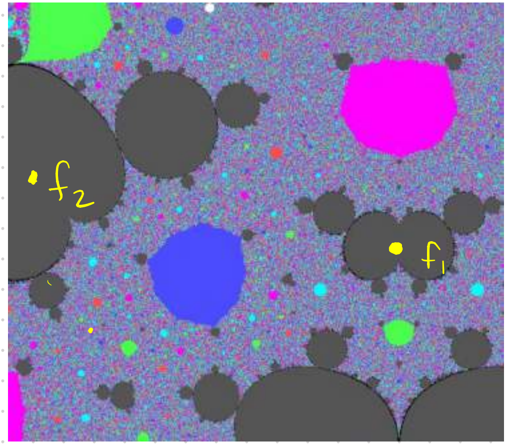
Combinatorics of $\mathcal{M}$, aka polynomials,

How much can we leverage that?

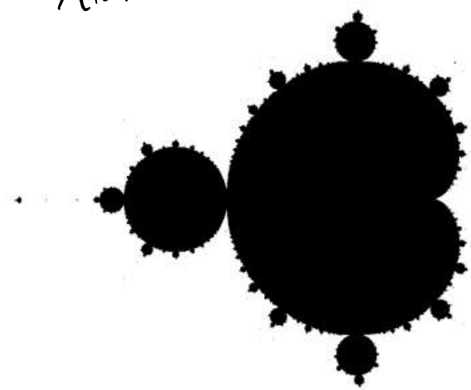
↳ is $\mathcal{M}_n$ (or $\partial \mathcal{M}_n = \mathrm{Bifur}_n$) connected?

The Program / Work-in-progress / Conjectures

1. Thm-in-Progress (D.)
   Per_n is "combinatorially connected"
2. Conjecture: "Veins exist"
   think: mating is cts along veins
3. Cor 1 + 2: Per_n "connected"
4. Conjecture: Every "non-mating" hyperbolic PCF map has a tuning that is a mating $f \sqcup g_0$ such that Θ "lands on" the Hubbard tree T_f
5. Thm-in-Progress (D.) $\exists^\infty$ such maps as in 4
6. Cor (2 + 4): M_n is connected
7. Cor (4 + ε): Good regions of Per_n admit nice models

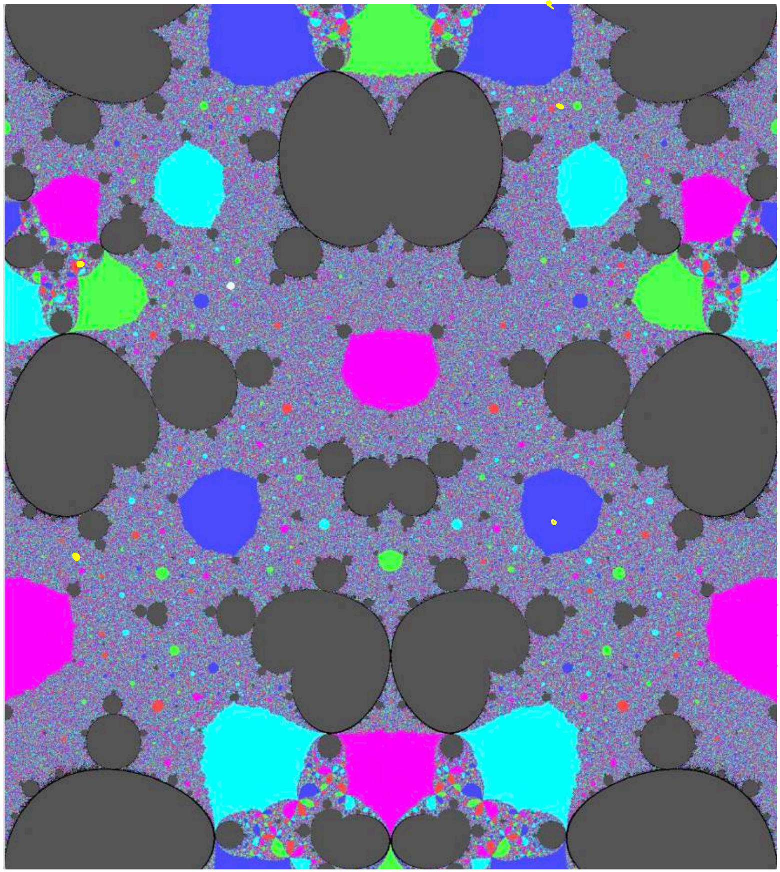


Alan Norton



What this talk hopes to do:

1. Convince you previous slide's hopes are reasonable
2. Convince you M_n combinatorics are really neat!



Per_5

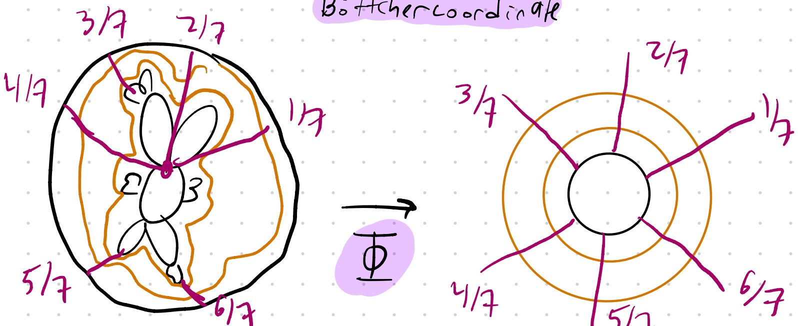
photo creds to Laurent Bartholdi!

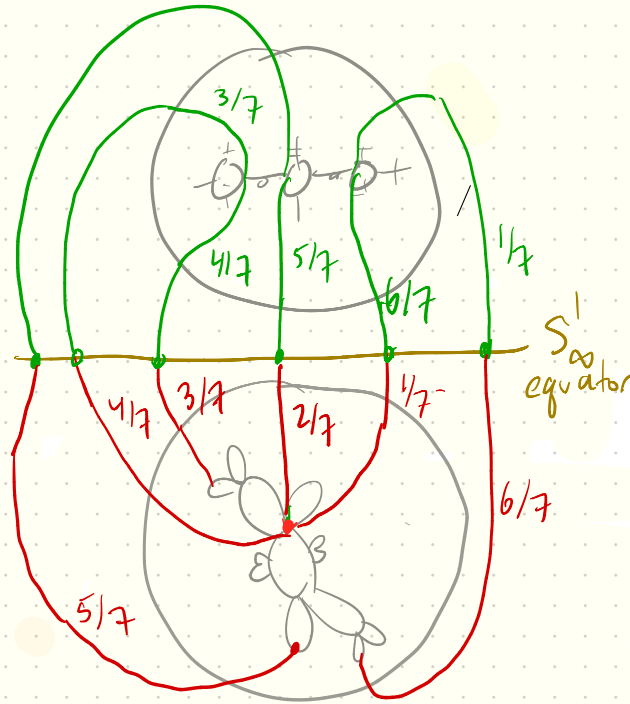
II] Setup: Matings, or how to make the next-simplest class of rat'l maps

Key: If f is locally connected:

$f \circlearrowright J_f \xrightarrow{\Phi} \overline{\mathbb{D}} \circlearrowright z^2$

Böttcher coordinate





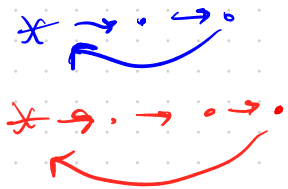
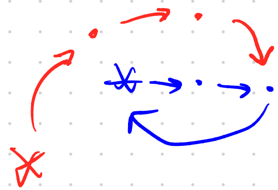
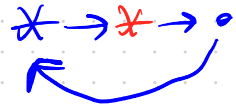
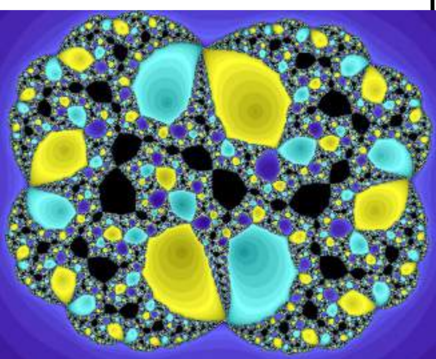
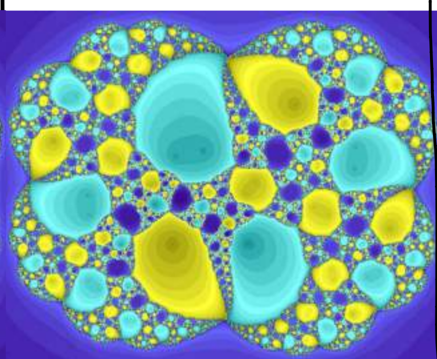
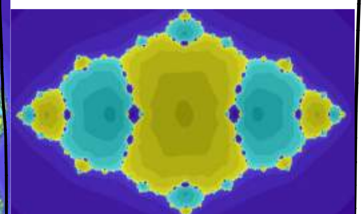
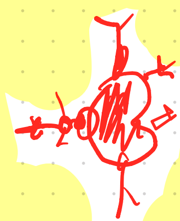
To mate f & g:

Glue J_f and J_g along (conjugate) external rays

Chéritat's slow mating videos!!!

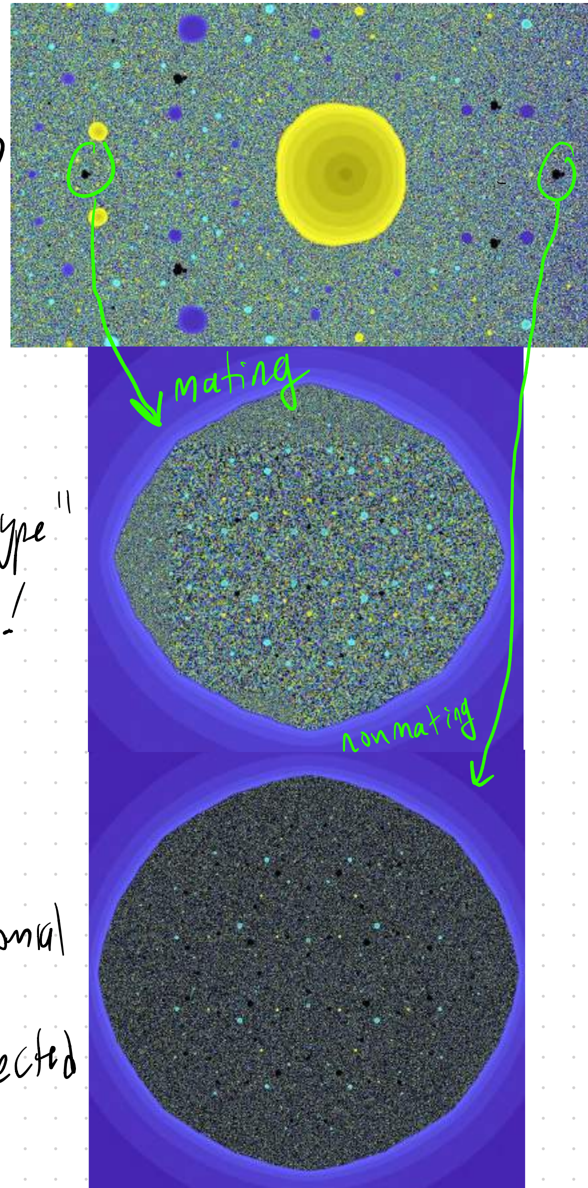
II] Set up: Milnor classification of Q.R.M. hyp. components

Cf. Geometry & Dynamics of QRM

Disjoint	Capture type	Bitransitive	Escape
			
$\bigcup f^n(*) \cap$ $* \to \cdot \to \cdot$ $= \emptyset$	free critical point tends to perscribed n-cycle	both critical points in same cycle	free critical pt tends to superattracting fixed pt
eg, matings!	⚠ Notation Warning! ⚠ Let's say "Type C"	yes, in my Per$_n$!	only in Per$_1$!
			E ↓  shift locus, cantor sets

Mating Locus $ML \subsetneq M_n \subsetneq Per_n$

$ML := \{$ maps in Per_n which are matings $f \sqcup g$ with f having a superattracting n-cycle $\}$

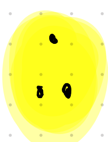


① In general, $f \sqcup g \neq g \sqcup f$

② For $n \geq 3$, there are "disjoint-type" maps that are non-matings!

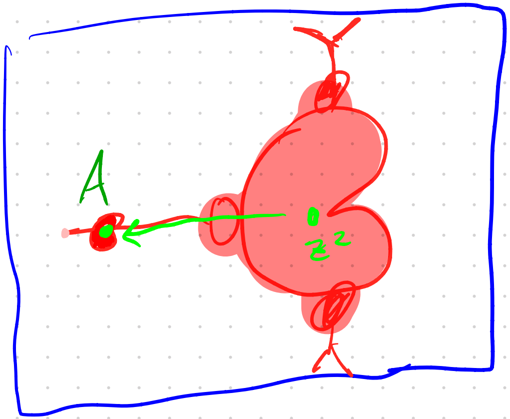
ie, ML does not even comprise all the PCF maps in Per_n

③ Fact (Milnor): every connected component of Per_n contains a polynomial

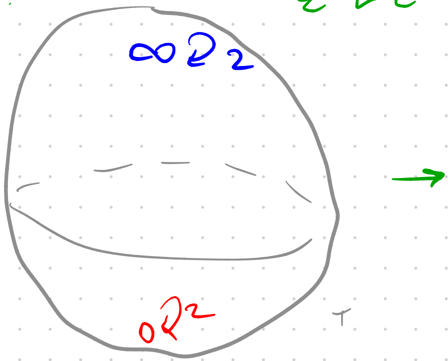
④  ML connected $\Rightarrow$ Per_n connected

Another way to think of matings

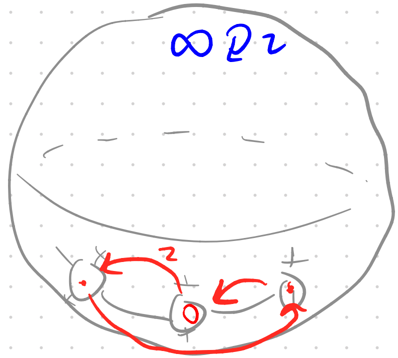
Per_1

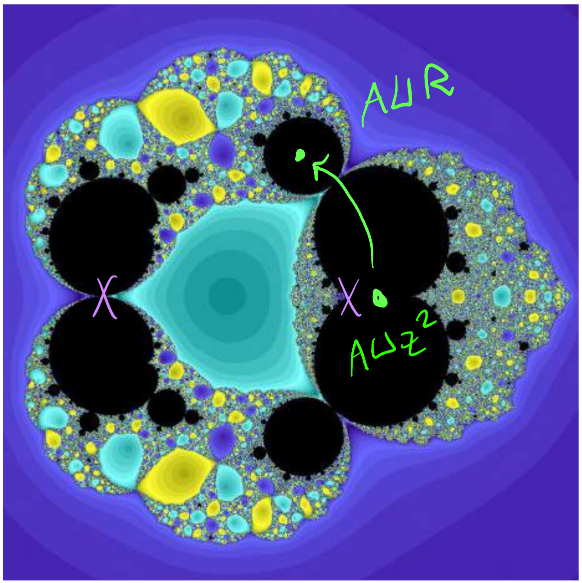


$z^2 \sqcup z^2$



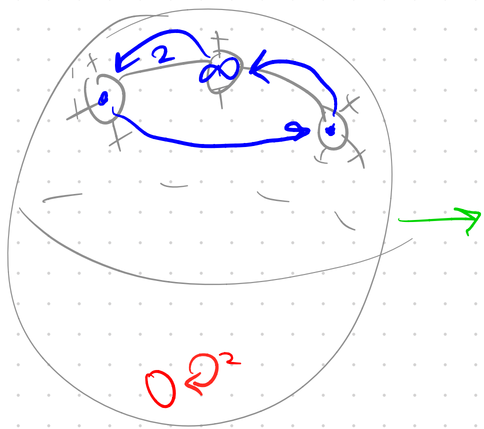
$z^2 \sqcup A$



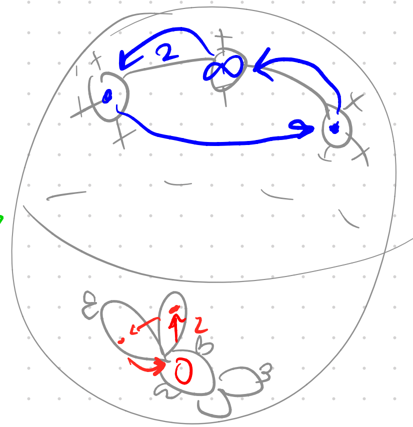


$A \sqcup z^2$

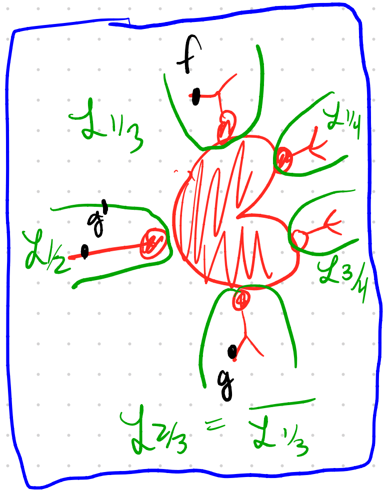
Per_3



$A \sqcup R$



Tan Lei – Shishikura – Rees Mating Theorem



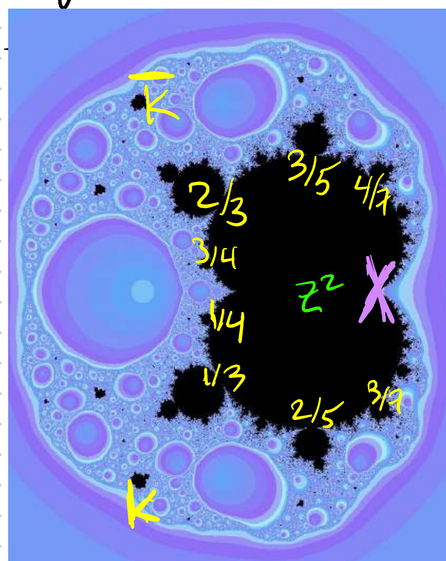
$\boxed{\text{Thm: } f, g \text{ in “non-conjugate limbs”} \iff f \sqcup g \text{ is comb. equivalent to a rat'l map}}$

Interpretation: f a polynomial with a critical n-cycle with $f \in \mathcal{L}_{p/q}$ for some p/q.

For fixed f & varying g, $f \sqcup g$ gives...

1. A puncture in Per_n according to g the fat $(1-p/q)$ rabbit

2. An entire "Mandelbrot's worth" of every other parameter $g \notin \mathcal{L}_{1-p/q}$, arranged as in the Mandelbrot set

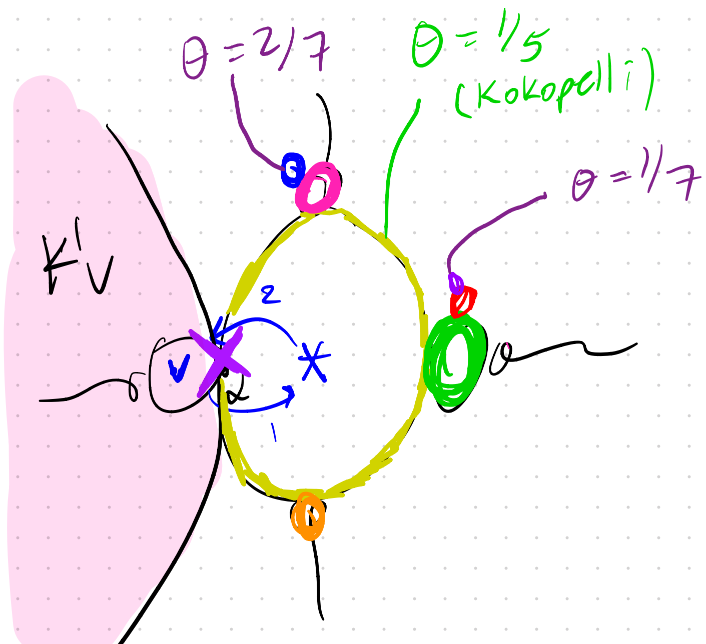


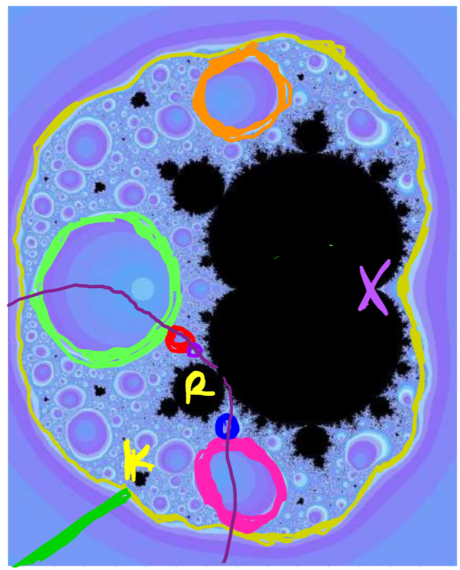
III] organizing the mating locus

aka: justifying the Mandelbrot "survives" the mating process

Per$_2$: Wittner, J. Luo, Dudko: $\mathcal{M}_2 \simeq (\mathcal{M} - \mathcal{L}_{1/2}) \sqcup (J_B - K'_v)$

key idea: use (pre)periodic points in $J_B - K'_v$ to parameterize $\mathcal{M}_2$



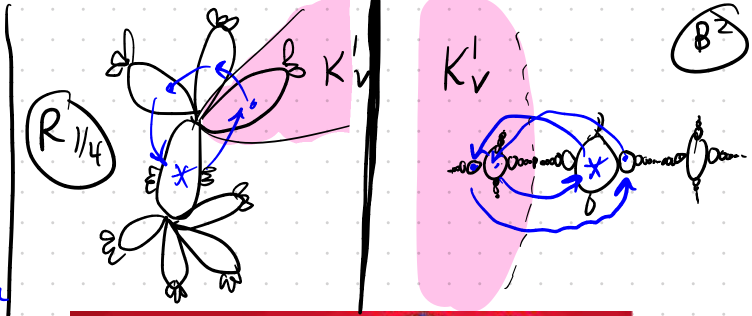


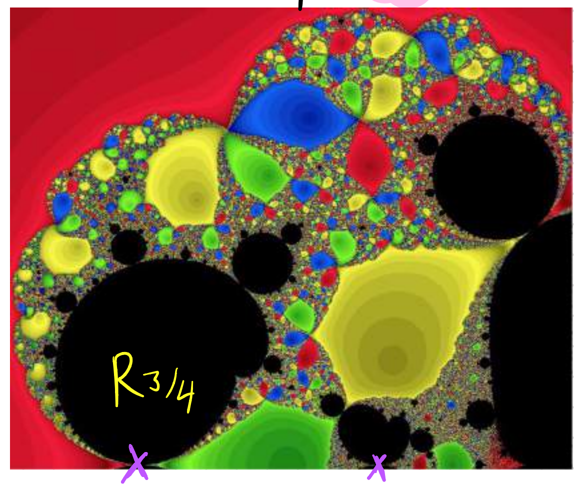
Bubble Rays:

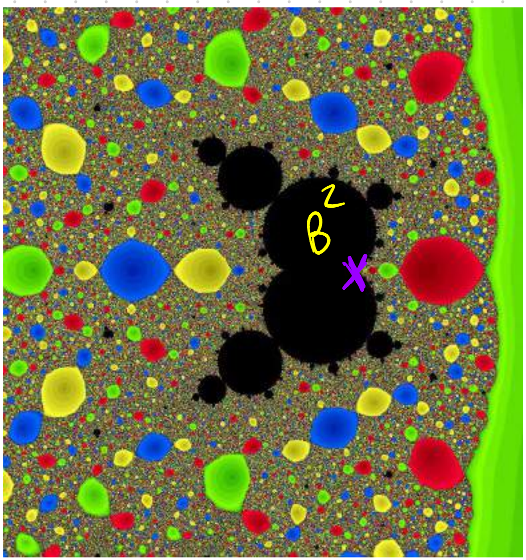
① Argument for $\mathcal{M}_2$ = Matings with Basilica

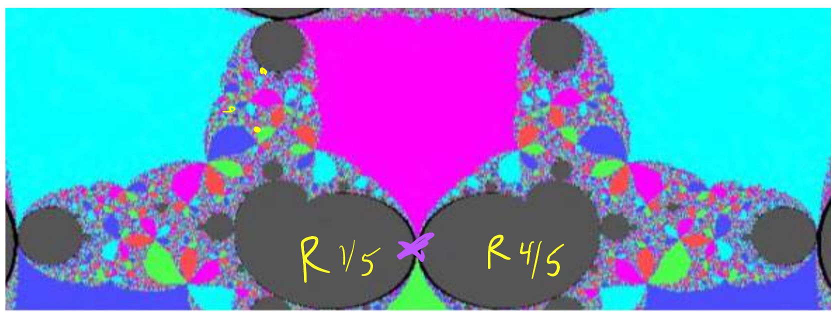
naturally works for "matings-with-MM" $\subset \mathcal{M}_n$

② J. Luo, Dudko actually construct puzzles ⇝ veins/Mandelbrot



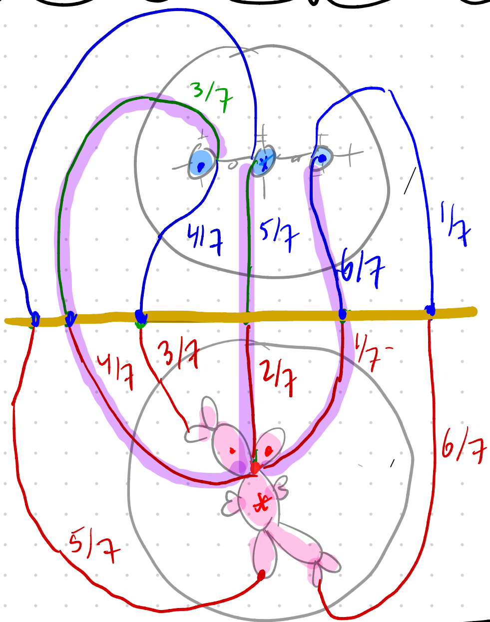




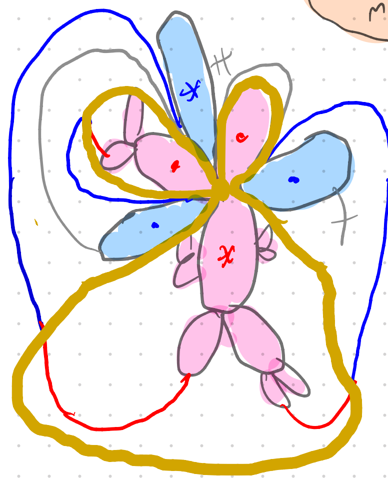


Shared Mating & the Wittner Flip:

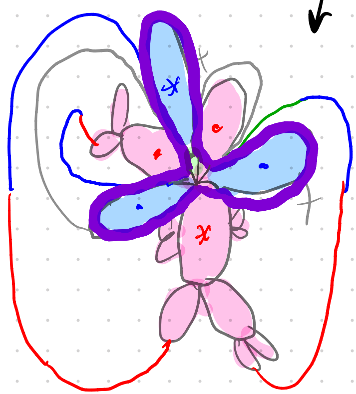
Recall: shared mating = unmate along many equators



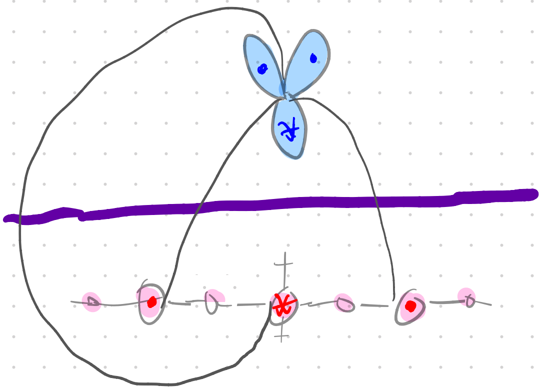
collapse along rays



find new equator



"unmate"



$A \sqcup R = R \sqcup A$

Shared Mating & the Wittner Flip:

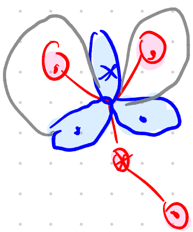
① All we used on previous slide was that

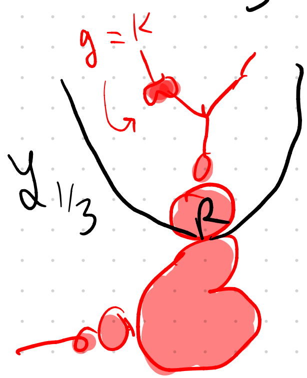
ⓐ f = Airplane

ⓑ f had "critical" external rays landing on α-fp of g = Rabbit

② So really g could be anywhere in $\mathcal{L}_{1/3}$

$A \sqcup K$:





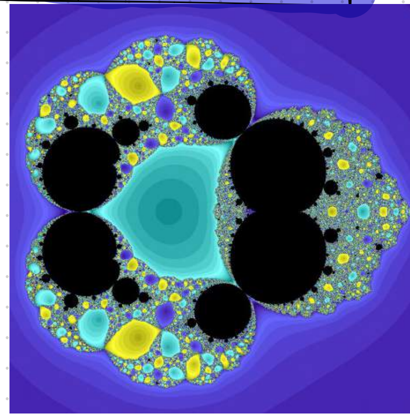
③ <u>Wittner Flip</u>

Let R be a p/q-Rabbit and f be a q-cycle with "critical external rays landing on α-fp of R"

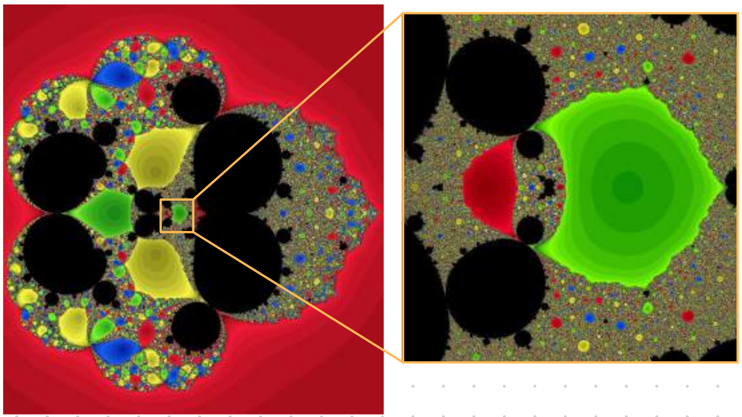
Then $\forall\, g \in \mathcal{L}_{p/q}$, $\exists\, h \in M$:

$$f \sqcup g = R \sqcup h$$

Parameter Space Interpretation of the Wittner Flip:

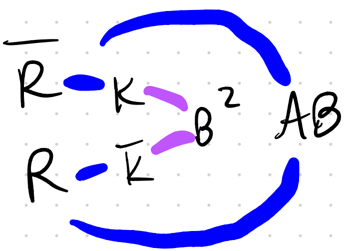


① $\frac{p}{q}$-Rabbit mating locus is connected to adjacent mating loci $\Rightarrow$ Per_3 connected



② Thm (D.)

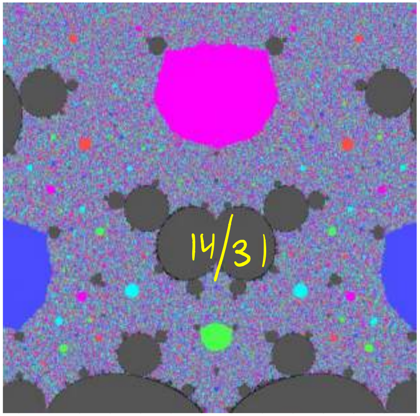
There is a generalized Wittner Flip for B^2



③ Cor: Per_4 is connected

Wittner Flips do not suffice for connectivity

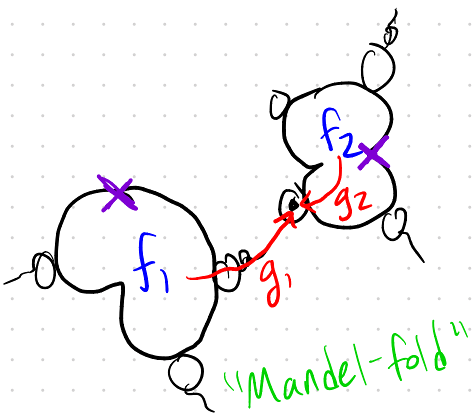
① For $n=5$, $\exists$ a polynomial whose mating locus is not (directly) connected to a rabbit



② General Situation: "primitive" shared matings

say f_1, f_2 are two polynomials with a critical n-cycle & suppose $\exists\, g_1, g_2$ such that

$$\boxed{f_1 \cup g_1 = f_2 \cup g_2}$$



③ Upshot: (if the veins exist)

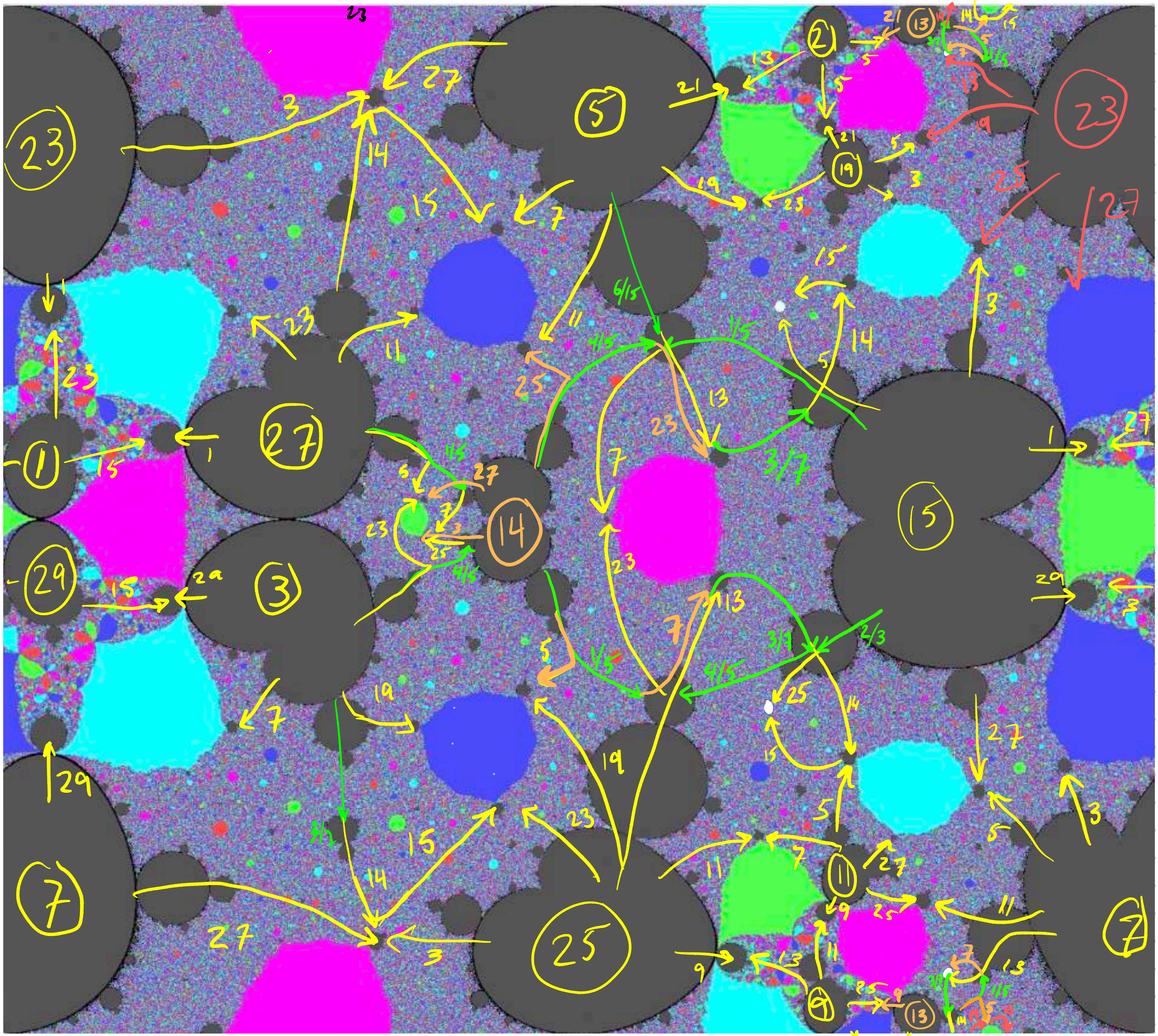
f_1 & f_2 are in the same connected component

④ Combinatorial Connectivity $\Leftarrow \exists$ enough shared matings (yes, $n \leq 7$)

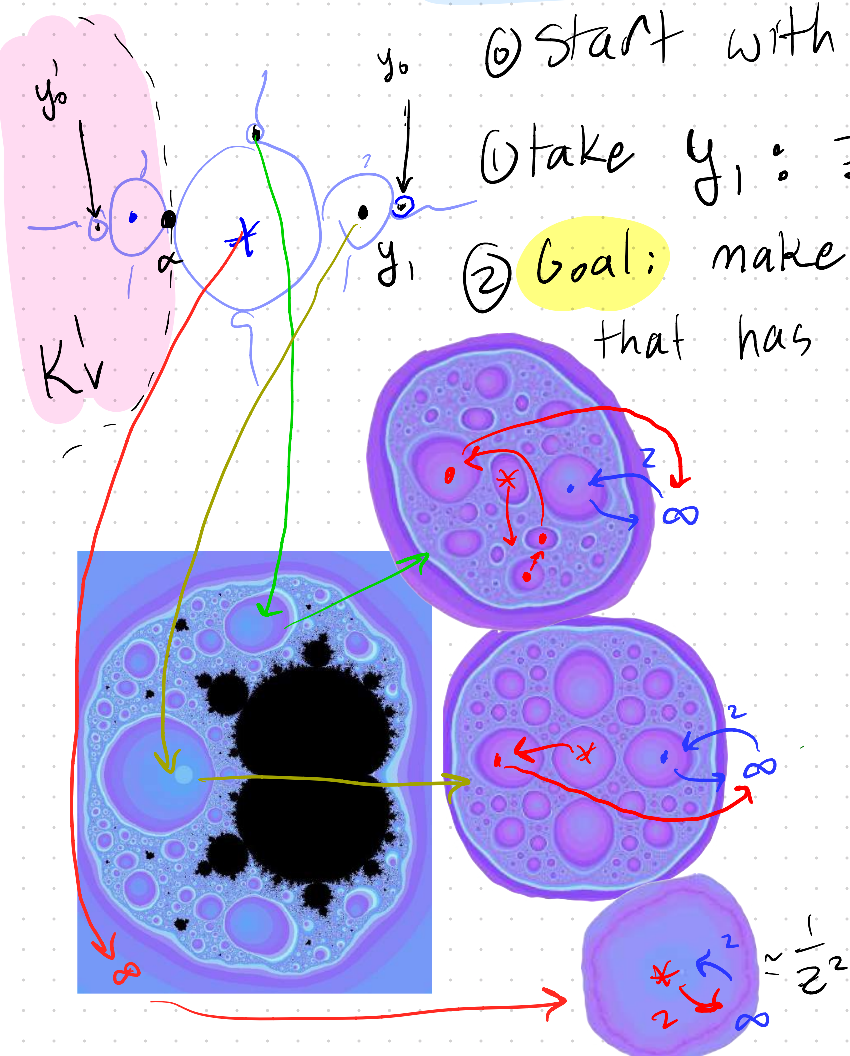
Where veins "ought to be" in Per_5

All #'s are angles

If no denominator, implicitly it's $\frac{\cdot}{31}$



Captures: Another way polynomials create rat'l dynamics



⓪ start with f polynomial w/ critical n-cycle

① take y_1: $\exists k$ with $f^k(y_1)$ in the n-cycle

② **Goal:** make new map, a capture at y_1,

that has $\begin{cases} y_1 \text{ as a critical value} \\ \text{forward orbit of } y \text{ the same} \end{cases}$

So: All captures of a polynomial give a $\boxed{\text{Type C}}$ or a $\boxed{\text{Bitransitive}}$ depending on whether y_1 is periodic or not

Reading Global Topology from Polynomial Combinatorics

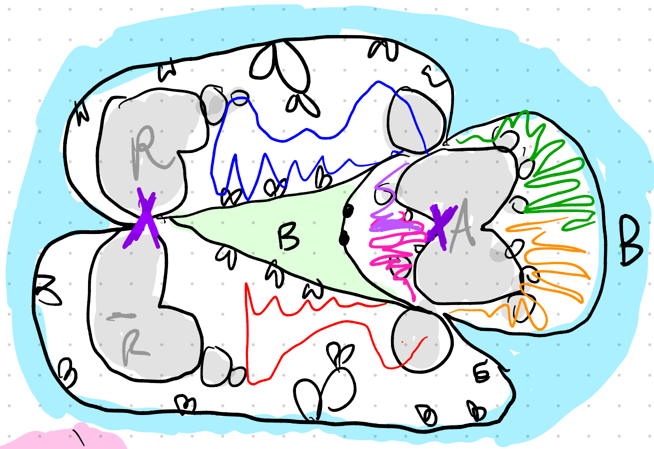
Are...	...All of...	Shared?
... Matings	... disjoint type? (aka PCF M_n) ✗	tells us about connectivity & genus
... Immediate Captures	... bitransitives? ?	
... Obstructed Matings	... punctures? ✓ Kiwi personal communications	tells us about number of punctures

Peterson, Epstein, Uhre, Milnor, Rees

Where the fun really begins (and the progress becomes "work in")

IV Primitive Combinatorics, NonMatings & Good Models

① What happens if we apply the model from Per_2 to the air lane locus?



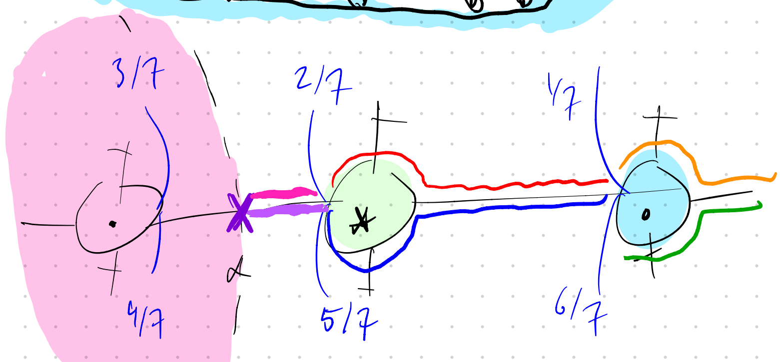
$\theta \in [1/7, 2/7]$

$\Longleftrightarrow$

$g_\theta \in \mathcal{L}_{1/3}$

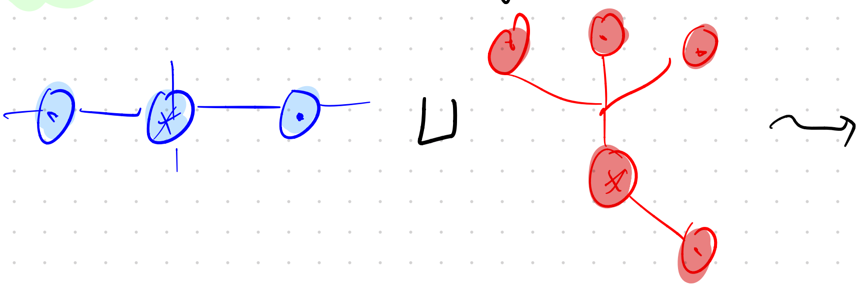
$\Longleftrightarrow$

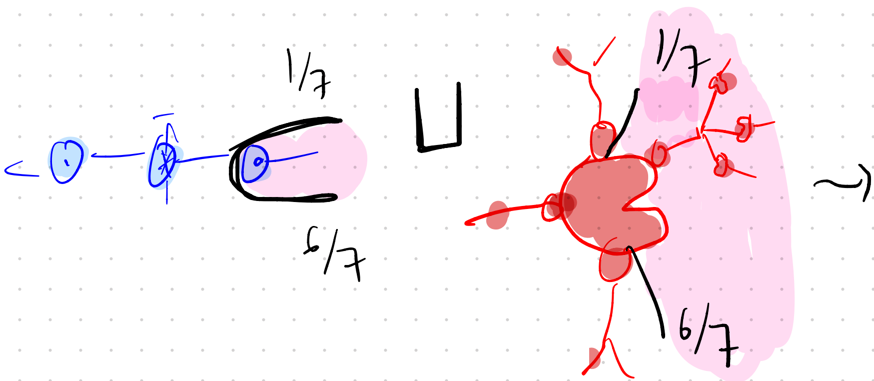
Wittner Flip

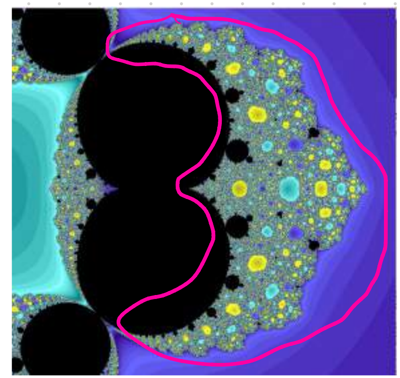


② Models (for "good" regions of Per_n) around primitives

ⓐ think: slow mating of 2 primitives

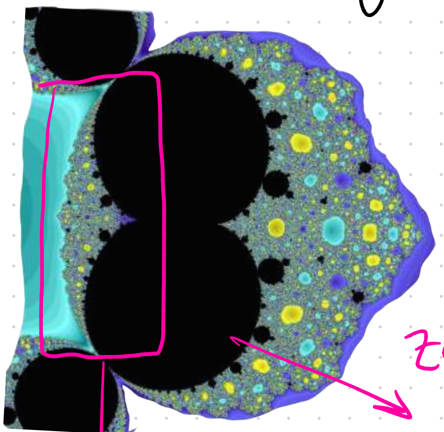




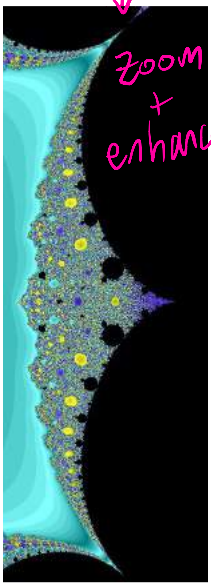


ⓑ General Hope for $\mathrm{Per}_n = \Sigma/\sim$? Using "QML"'s?

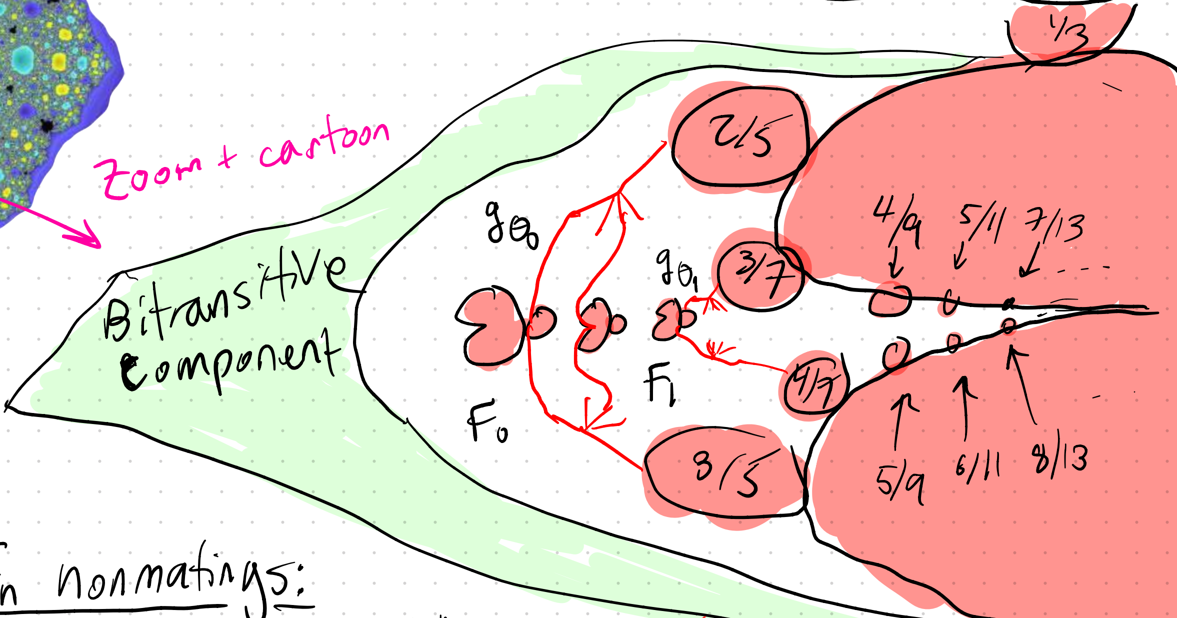
③ Non-matings, cf.



Zoom + cartoon



Conjecture Every "non-mating" hyperbolic PCF map has a tuning that is a mating $f \sqcup g_\Theta$ such that Θ "lands on" the Hubbard tree T_f



$\exists^\infty F_n$ nonmatings:

- F_0 "Wittner's example"
- period $= q_{n-1}+1$
- $F_n \vDash B = A \sqcup g_{\Theta_n}$
- $\Theta_n = \dfrac{(4^n-1)}{3(4^n+1)}$