Trace Ideals and Applications

Haydee Lindo, Harvey Mudd College

Fellowship of the Ring

March 15 2022.

"Recent" Developments

Auslander & Goldman:

: "Maximal Orders" 1960

Appendix: all about trace ideals

What is a Trace Ideal?

Def'n Given R-modules M the trace ideal of M is:

$$\tau_M(R) := \sum_{\alpha \in \mathrm{Hom}(M,R)} \alpha(M)$$

Ex $\tau_{(x,y)}(\mathbb{Q}[\![x,y]\!]) = (x,y)$

Ex $R \subseteq S \subseteq \overline{R}$ $\quad \tau_S(R) =$ conductor of S in R.

Every module has a trace ideal

The trace ideal is an invariant of the module.

Q: What does $\tau_M(R)$ tell us about M?

Etymology

$M^* := \mathrm{Hom}_R(M, R)$.

$\mathrm{End}_R(R^n) := \mathrm{Hom}_R(R^n, R^n)$ = $n \times n$ matrices w/ entries in R.

projective

$$R^n \otimes_R (R^n)^* \cong \mathrm{End}_R(R^n) \xrightarrow[\text{sum along diagonal.}]{\text{trace}} R$$

$$m \otimes \alpha \longmapsto \alpha(-)m$$

$$\begin{aligned} \alpha(-)m : \mathbb{R}^n &\longrightarrow \mathbb{R}^n \\ x &\longmapsto \alpha(x)m \end{aligned}$$

Etymology Cont'd

$$\begin{tikzcd}
R^n \otimes (R^n)^* \arrow[r, "\cong"] \arrow[rr, bend left, dashed, "\text{"trace"}"] & \mathrm{End}(\mathbb{R}^n) \arrow[r, "\text{trace}"] & R
\end{tikzcd}$$

$$m \otimes \alpha \xmapsto{\text{ev.}} \alpha(m)$$

Note: this map exists even when M is not projective

$\vartheta_M : M \otimes M^* \longrightarrow R$ is the trace map

$$m \otimes \alpha \longmapsto \alpha(m)$$

$$\operatorname{Im} \vartheta_M = \tau_M(R)$$

What are Trace Modules?

Def'n Given R-modules M & X the trace of M in X is:

$$\tau_M(X) := \sum_{\alpha \in \mathrm{Hom}(M,X)} \alpha(M)$$

$$\tau_M(X) := \mathrm{Im}\left(ev : M \otimes_R \mathrm{Hom}_R(M,X) \longrightarrow X \right).$$

$$m \otimes \alpha \longmapsto \alpha(m).$$

Ex $(R, \mathfrak{m}, k)$ local ring $\quad \tau_k(M) = \mathrm{socle}(M)$

Given any 2 modules there exists 2 trace modules

Def'n: A module M is called a trace module provided $\exists$ modules A & X s.t. $M = \tau_A(X) \subseteq X$

an ideal I is called a trace ideal if $\exists$ A s.t. $I = \tau_A(R) \subseteq R$.

Q: What does "being a trace module" tell us about a module?

A: Not much because:

every module is a trace module

$\tau_M(M) = M, \quad \tau_R(M) = M, \quad \tau_G(M) = M.$

for any generator G in $\mathrm{Mod}\, R$

Proper Trace Module

Def'n An R-module M is called a proper trace module provided $\exists$ R-modules A & X st.

$$M = \tau_A(X) \subsetneq X.$$

Q: What does being a "proper trace module" say about a module?

Trace Ideals are Calculable (Vasconcelos 1991)

M finitely presented with presentation matrix $[M]$

$$\begin{tikzcd} R^n \arrow[r, "{[M]}"] & R^n \arrow[r] \arrow[d] & 0 \\ & M \arrow[ur] & \end{tikzcd} \xRightarrow{\mathrm{Hom}(-,R)} \begin{tikzcd} R^t \arrow[rr, "{[X]}"] \arrow[dr] & & R^n \arrow[r, "{[M]^T}"] & R^m \\ & M^* \arrow[ur] & & \end{tikzcd}$$

$\tau_M(R)$ is generated by the entries of $[X]$

$\tau_M(R) = I_1(\text{left Kernel of } [M])$

Example

$$S = \frac{\mathbb{R}[x,y,z]}{(xy - z^2)} \qquad I = (x,z)$$

$$\begin{tikzcd}
S^2 \arrow[r, "{\begin{bmatrix} -z & y \\ x & -z \end{bmatrix} = [I]}"] & S^2 \arrow[r] \arrow[dr] & 0 \\
 & & (x,y) \arrow[u]
\end{tikzcd}$$

$$\tau_I(R) = I_1(\text{left kernel of } [I])$$

$$= I_1\left(\begin{bmatrix} x & z \\ z & y \end{bmatrix}\right)$$

$$= (x, y, z)$$

Elements vs Homomorphisms

Prime ideals cannot be multiplied into

$\mathfrak{p} \in \operatorname{Spec} R.$ $\quad x, y \in R$

$$xy \in \mathfrak{p} \Rightarrow x \in \mathfrak{p} \text{ or } y \in \mathfrak{p}.$$

Trace ideals cannot be mapped out of:

I a trace ideal & $\alpha \in \operatorname{Hom}(I, R)$

then $\alpha \in \operatorname{Hom}(I, I)$, i.e. $\operatorname{Im} \alpha \subseteq \tau_I(R)$

Properties of Trace Ideals

1. Easy to Calculate (Macaulay 2)
2. Easy to characterize: Inclusion $\tau_M(R) \subseteq R$ induces an isomorphism:

$$\tau_M(R)^* = \mathrm{End}_R(\tau_M(R))$$

Idea:
$$\begin{tikzcd} M^{(\Lambda)} \arrow[r] \arrow[rr, bend right, dashed] & \tau_M(R) \arrow[r] & R \end{tikzcd}$$

Easy to test:

$I = \tau_M(R)$ iff $\tau_I(R) = I$

Properties of trace ideals

(3) "Closed" : $\tau_{\tau_M(R)}(R) = \tau_M(R)$

($\supseteq$) $I \subseteq \tau_I(R) \quad \forall I$

($\subseteq$) (2)
$$\begin{tikzcd} M^{(\Lambda)} \arrow[r, two heads] \arrow[rr, dashed, bend right] & \tau_M(R) \arrow[r] & R \end{tikzcd}$$

(see also: Epstein, Perez, R.G.)

Properties of trace ideals

4. Ubiquitous: $0 \to I \to R \to R/I \to 0$ $\quad \mathrm{Hom}(-, R)$

$$0 \to \mathrm{Hom}(R/I, R) \to \mathrm{Hom}(R, R) \xrightarrow{\text{surj}} \mathrm{Hom}(I, R) \to \cancel{\mathrm{Ext}^1_R(R/I, R)} \to 0$$

$r \longmapsto r$

R self inj or $\mathrm{grade}(I) \geq 2$.

$\Rightarrow \alpha(I) \subseteq I \;\; \forall \alpha \in I^*$

$\Rightarrow I = \tau_I(R)$.

Properties of Trace Ideals

(5) Reflects properties of M:

$(R, \mathfrak{m}, k)$ local, $\tau_M(R) = R$ iff $M \cong N \oplus R$

non local case: $\tau_M(R) = R$ iff M is a generator in $\operatorname{Mod} R$.

(6) Behaves well under flat extensions:

$$(\tau_M(R))_{\mathfrak{p}} = \tau_{M_{\mathfrak{p}}}(R_{\mathfrak{p}}), \quad \widehat{\tau_M(R)} = \tau_{\hat{M}}(\hat{R}).$$

Applications:

R has canonical module ω

R Gorenstein iff $\omega \cong R$ iff $\tau_\omega(R) = R$

$\Rightarrow \tau_\omega(R)$ is a measure of Gorensteinness of R.

(See work of: Herzog, Hibi, Stamate, Ding).

7. $M \cong N \Rightarrow \tau_M(R) = \tau_N(R)$

$\tau_M(R) \cong \tau_N(R) \Rightarrow \tau_M(R) = \tau_N(R)$

(turns isomorphisms into equalities.)

8. We say M generates N if $\exists\ \Lambda$ s.t.

$$M^{(\Lambda)} \twoheadrightarrow N$$

M gens $N \Rightarrow \tau_M(N) = N$ and $\tau_M(R) \supseteq \tau_N(R)$

$\tau_M(R) \twoheadrightarrow \tau_N(R) \Rightarrow \tau_M(R) \supseteq \tau_N(R)$.

(turns surjections into containments)

Sharing Traces

Lemma: $\tau_M(R) = \tau_{M \otimes M^*}(R)$

Pf// M generates $M \otimes M^*$ and

$M \otimes M^*$ generates $\tau_M R$

$$\tau_M(R) \supseteq \tau_{M \otimes M^*}(R) \supseteq \tau_{\tau_M(R)}(R) = \tau_M(R). \quad \blacksquare$$

(See also: Herbera, Prihoda)

Related Ideas

Order Ideals: $m \in M$. $\quad M^*(m) = \sum_{\alpha \in \mathrm{Hom}(M,R)} \alpha(m)$

If $\{m_1, \dots, m_n\}$ generate M then.

$$(M^n)^*((m_1, \dots, m_n)) = \tau_M(R).$$

(See work of Eisenbud, Huneke, Ulrich.)

$$\{\text{Order Ideals}\} \supseteq \{\text{Trace Ideals}\}$$

Related Ideas:

$$
\begin{tikzcd}
 & M \arrow[ddl, dash] \arrow[ddr, dash] \arrow[rr, dash, dashed] & & T_M(k) \arrow[dr, dash, dashed] & \\
 & & & & Z(\mathrm{End}(M)) \\
M \otimes M^* \arrow[rr, dash] \arrow[uurrr, dash, dashed] & & \mathrm{End}(M) \arrow[uur, dash, dashed] \arrow[urr, dash, dashed] & &
\end{tikzcd}
$$

Applications of $\tau_M(R)$

$$M \otimes M^* \xrightarrow{ev} \tau_M(R) \longrightarrow 0$$

Apply $\mathrm{Hom}(\_, R)$

$$\begin{tikzcd}
0 \arrow[r] & \mathrm{Hom}(\tau_M(R), R) \arrow[rr] & & \mathrm{Hom}(M \otimes M^*, R) \arrow[d, "\cong \text{ Hom - Tensor adj}"] \\
& & & \mathrm{Hom}(M, M^{**}) \arrow[d, "\cong \text{ } M \text{ reflexive}"] \\
& \mathrm{Hom}(\tau_M(R), \tau_M(R)) \arrow[uu, "="] \arrow[rr, hook, dashed, "\sigma"] & & \mathrm{Hom}(M, M)
\end{tikzcd}$$

Trace Ideals and Endomorphism Rings

Thm (L - 2017) M reflexive and faithful

$$\operatorname{End}_R(\tau_M(R)) \xrightarrow[\cong]{\sigma} Z(\operatorname{End}(M))$$

is an isomorphism of R-algebras

$$Z(\operatorname{End}(M)) := \{ f \in \operatorname{End}(M) \mid fg = gf \ \forall g \in \operatorname{End}(M) \}.$$

Uses Yasutaka Suzuki σ^{-1} 1974 ("Double Centralizers of torsionless modules")

Ex Apply to $M_n(\mathbb{R})$?

$$M_n(\mathbb{R}) = \mathrm{End}(\mathbb{R}^n)$$

$$\mathcal{T}_{\mathbb{R}^n}(\mathbb{R}) = \mathbb{R}.$$

$$Z(\mathrm{End}(\mathbb{R}^n)) \overset{\text{Thm}}{\cong} \mathrm{End}(\mathbb{R}) \cong \mathbb{R}$$

$$r \cdot \begin{bmatrix} 1 & & 0 \\ & \ddots & \\ 0 & & 1 \end{bmatrix}_{n \times n} \longleftarrow\!\shortmid \; r \cdot 1$$

$n \times n$ matrices that commute with the others : $\left\{ \begin{bmatrix} r & & 0 \\ & \ddots & \\ 0 & & r \end{bmatrix}_{n \times n} \middle|\ r \in \mathbb{R} \right\}$

When $\mathcal{T}_M(R)$ contains a nonzerodivisor $\operatorname{End}(\mathcal{T}_M(R)) \subseteq Q(R)$.

$\mathcal{T}_M(R)$ determines the largest subring of $Q(R)$ over which M is a module (see also Faber).

<u>Thm</u> (above). $\operatorname{End}(\mathcal{T}_M(R)) = Z(\operatorname{End}(M)) \subseteq Q(R)$

w/o reflexive assumption:

$Z(\operatorname{End}(M^*)) = \operatorname{End}(\mathcal{T}_{M^*}(R)) = \operatorname{End}(\mathcal{T}_M(R)) \subseteq Q(R)$

Consequences:

Recall :

Thm (Vasconcelos 1967)

R 1-dim local Gorenstein ring

Then $\mathrm{End}_R(M)$ is free iff M is free

Thm (L - 2017)

R local commutative Noetherian w/ depth $R \leq 1$

M reflexive

Then $\mathrm{End}(M)$ has a free summand iff M does

& $\mathrm{End}(M)$ is R-free iff M is.

Pf// (Sketch) $\tau_M(R)^* = \mathrm{End}_R(\tau_M(R))$

$= \mathrm{End}_R(\tau_{M\otimes M^*}(R))$ $\Big) = Z(\mathrm{End}(M\otimes M^*))$.

$\overset{\text{Thm}}{=} \mathrm{End}_R(\tau_{(M\otimes M^*)^{**}}(R))$ $\Big)$ M reflexive

$\overset{\text{refl}}{=} \mathrm{End}_R(\tau_{\mathrm{End}(M)}(R))$

$= R$ $\Big)$ End(M) free summand.

depth $R \le 1$ & $(\tau_M(R))^* = R \overset{\text{Lemma}}{\Rightarrow} \tau_M(R) = R$

$\Rightarrow M \cong N \oplus R$ $\blacksquare$

Individual $\tau_M(R)$ vs $\{\tau_M(R)\}$

So far: How individual trace ideals help us understand M, $Z(\mathrm{End}(M))$ & $\mathrm{End}(M)$

Next: Reasons to study $\{\tau_M(R) \mid M \in \mathrm{Mod}\, R\}$.

Rigidity Conjecture: HWC.

Huneke - Wiegand Conjecture (HWC, 1997)

R 1-dim Gorenstein domain

M reflexive

$\mathrm{Ext}^1_R(M,M) = 0$ (equiv $M \otimes_R M^*$ torsionfree).

Then M free.

Lemma HWC is true for all ideals isomorphic to a trace ideal.

Pf. $C := \mathrm{End}(\tau_M(R))$

$$\tau_M(R) \otimes_R \tau_M(R)^* \underset{\text{trace}}{\overset{\text{t.f.}}{=}} \tau_M(R) \otimes_C C \cong \tau_M(R)$$

$\Rightarrow (\tau_M(R))^*$ 1-generated $/R$

$\Rightarrow (\tau_M(R))^* = R$

refl. $\Rightarrow \tau_M(R) = R$ (see also: Huneke, Iyengar, Wiegand)

Questions

Q: Which ideals are isomorphic to trace ideals?

Q: How big is $T(R) = \{\tau_M(R)\}_{M \in R\text{-Mod}}$?

How big is the class of proper trace modules?

Q: What is $T(R)$ when R is Gorenstein?

Q: What is the relationship between trace ideals and self extensions?

Rigidity

Defn M is **rigid** if $\mathrm{Ext}^1_R(M,M)=0$

Ex $R = k[\![x,y]\!]/(xy)$ $\quad M = R/(x) \cong (y)$

free res:

$$\begin{tikzcd}
\cdots \arrow[r, "\cdot x"] & R \arrow[r, "\cdot y"] & R \arrow[r, "\cdot x"] & R \arrow[rr] \arrow[dr] & & 0 \\
& & & & M \arrow[ur] &
\end{tikzcd}$$

$\mathrm{Hom}(-, R/(x))$

$$
\begin{tikzcd}
0 \arrow[r] & \mathrm{Hom}(R, R/(x)) \arrow[r, "\cdot x"] & \mathrm{Hom}(R, R/(x)) \arrow[r, "\cdot y"] \arrow[d, "\cong"] & \mathrm{Hom}(R, R/(x)) \arrow[d, "\cong"] \\
& & R/(x) \arrow[r, dashed, "\cdot y"] & R/(x)
\end{tikzcd}
$$

$\mathrm{Ext}^1_R(R/(x), R/(x)) = 0$

$R/(x)$ <u>not</u> free $\because$ $\mathrm{Ann}_R\, R/(x) = (x)$

Interest in Rigid Modules

Generalized Nakayama's Conjecture (1975)

Λ Artin Algebra

If I a Λ-module, injective indecomposable

$\Rightarrow$ I appears as a direct summand in the minimal injective resolution of Λ.

Equivalently

Auslander–Reiten Conjecture (Auslander–Reiten 1975)

Λ Artin Algebra

$M \in \Lambda\text{-mod}$

Then $\mathrm{Ext}^i_\Lambda(M,M)=0=\mathrm{Ext}^i_\Lambda(M,\Lambda)$ $\forall\ i>0$

Then M projective.

Our Setting: $(R, \mathfrak{m})$ comm local Noeth.

$M \in R\text{-mod}$

ARC (Auslander - Solberg - Ding 1993)

If $\operatorname{Ext}^i_R(M,M) = 0 = \operatorname{Ext}^i_R(M,R) \quad \forall\ i > 0$

Then M free.

Behaviour of Rigid Modules

$(R, \mathfrak{m})$ 0-dim

$M \subsetneq X \in R\text{-mod}$

$$0 \to M \xrightarrow{\subseteq} X \xrightarrow{\pi} X/M \to 0$$

$\mathrm{Hom}(M, \_)$.

$$0 \to \mathrm{Hom}(M,M) \to \mathrm{Hom}(M,X) \xrightarrow[\twoheadrightarrow]{\pi_*} \underbrace{\mathrm{Hom}(M, X/M)}_{\neq 0} \longrightarrow \overbrace{\mathrm{Ext}^1_R(M,M)}^{0}$$

$$M \twoheadrightarrow R/\mathfrak{m} \hookrightarrow X/M.$$

Meaning :

$$\begin{tikzcd} M \arrow[rr, "\bar{\alpha} \neq 0"] \arrow[dr, dashed, "\exists \alpha"'] & \arrow[d, phantom, "\circlearrowleft"] & X/M \\ & X \arrow[ur, "\pi"'] & \end{tikzcd}$$

$\Rightarrow \ \exists \ \alpha \in \mathrm{Hom}(M,X)$ st $\mathrm{Im}\,\alpha \notin M.$

Idea: Rigidity is the "opposite" phenomenon to being trace.

Lemma: M proper submodule of X

If $\mathrm{Hom}_R(M, X/M) = 0$ then $M = \tau_M(R)$

If $M = \tau_M(X)$ & M rigid then $\mathrm{Hom}(M, X/M) = 0$.

Note: $\mathrm{Hom}(M, X/M) \neq 0$ for pairs (M, X)

- (I, R) I proper $\mathfrak{m}$-primary ideal
- (I, R) I contains a nzd
- (M, R) if X/M has finite length.

Consequence

Prop: R local Artinian Gorenstein

M a positive or negative syzygy of a.

proper trace module

then $\mathrm{Ext}^1_R(M, M) \neq 0$.

Q: How big is the class of proper trace modules?

Partial Answer:

Thm (–, Pande) R comm Noeth TFAE:

1) R is Artinian Gorenstein

2) Every ideal is a trace ideal.

Easy Corollary:

R local Artinian Gorenstein

ARC holds for positive and negative syzygies of ideals

Extensions

Thm (Kobayashi, Takahashi '19) "Rings whose ideals are isomorphic to trace ideals"

Every ideal of R is isomorphic to a trace ideal iff one of the following holds:

(dim 0): R Artinian Gorenstein

(dim 1): R is a hypersurface of Krull dimension 1 & mult ≤ 2

(dim ≥ 2): R is a UFD.

Arf Rings

Recall

Def'n $(R, \mathfrak{m}, k)$ local Noeth of Krull dim d

$I \subseteq R$ an $\mathfrak{m}$-primary ideal

M an R-module

the multiplicity of I on M is $e_R(I, M) := \lim\limits_{n \to \infty} \frac{d!}{n^d} \lambda_R(M/I^n M)$

length

the multiplicity of M is

$$e_R(M) := e_R(\mathfrak{m}, M)$$

the multiplicity of R is $e(R) := e_R(R)$

Note: When R is 1 dim Cohen-Macaulay multiplicity is an upper bound on the # of generators of any ideal $I \subseteq R$

Def'n R is said to have minimal multiplicity provided:

$$\underbrace{e(R)}_{\text{mult against } \mathfrak{m}} = \underbrace{\operatorname{embdim}(R)}_{\substack{\text{min \# of gens} \\ \text{of max ideals}}} - \underbrace{\dim R}_{\substack{\text{krull} \\ \text{dim}}} + 1$$

Blowups

Given a regular ideals I

contains nzd

$\exists$ filtration

$$R \subseteq \underset{Q(R)}{I : I} \subseteq \underset{Q(R)}{I^2 : I^2} \subseteq \cdots \subseteq \overline{R}$$

Def'n The Blowup of R at I

$$R^I := \bigcup_{n \geq 1} (I^n :_{Q(R)} I^n)$$

Note:

- R^I module finite over R
- If I has a $\underbrace{\text{principal reduction}}_{\exists\, n \text{ s.t. } (x)I^n = I^{n+1}}$ (x)

$$R^I = R\left[\frac{I}{x}\right]; \quad \frac{I}{x} := \{ a/x \mid a \in I \} \subseteq Q(R)$$

Stability

Def'n I is stable if $R^I = I : I$

Lemma (Lipman '71) "Stable Ideals and Arf Rings".

A regular ideal is stable iff $\exists\ x \in I$ satisfying the following equivalent conditions:

1) $I^2 = xI$
2) x regular & Ix^{-1} is a ring
3) x regular & $Ix^{-1} = R^I$

Def'n The integral closure $\overline{I}$ of an ideal I is the collection $x \in R$ s.t. $\exists$ $a_i \in I^i$ w/

$x \in R$ s.t. $\exists$ $a_i \in I^i$ w/

$$x^n + a_1 x^{n-1} + \dots + a_{n-1} x + a_n = 0$$

Def'n A 1-dim Cohen Macaulay ring R is called an Arf ring provided every integrally closed regular ideal is stable.

eg. Celikbas, Celikbas, Goto, Taniguchi
(Generalized Gorenstein Arf rings 2018)

k field

$R = k[[t^4, t^7, t^9, t^{10}]]$

$$R_1 := R^{\mathfrak{m}} = R\left[\frac{\mathfrak{m}}{t^4}\right] = k[[t^3, t^4, t^5]]$$

t^4 is a reduction of $\mathfrak{m}$

$t^4 R_1 = \mathfrak{m} \Rightarrow t^4 \mathfrak{m} = \mathfrak{m}^2$

$\mathfrak{m}_1 :=$ unique max ideal of R_1

$$R_2 := R_1^{m_1} = R\left[\frac{m}{t^s}\right] = k[[t]] = \overline{R}$$

$$m_1^2 = t^3 m_1$$

$\Rightarrow$ Distinct Blow ups $R = R_0, R_1, R_2 = \overline{R}$

$\Rightarrow$ R Arf $\because$ $m = m_0, m_1, m_2$ are stable.

Facts:

dim 1 $\Rightarrow$ $e(R)$ = min # gens required for high powers of $\mathfrak{m}$

$\mathfrak{m}$ stable $\Rightarrow$ $\mathfrak{m}^2 \cong \mathfrak{m}$

$\Rightarrow$ all powers of $\mathfrak{m}$ have same # of gens

$\Rightarrow$ $\operatorname{embdim}(R) = e(R)$.

Thm (CCGT 2.7)

R 1-dim Cohen Macaulay local w/ canonical module.

TFAE:

1) R Arf

2) $\operatorname{embdim}((R_n)_\mathfrak{m}) = e((R_n)_\mathfrak{m})$ $\forall$ $n \geq 0$

get from stability of $\mathfrak{m}$.

& $\mathfrak{m}$ max ideal of R_n.

eg 1 dim hypersurfaces of mult ≤ 2

$k[[x,y]]/(y^2-x^3)$

Thm (Sally Vasconcelos 1974) "Stable Rings"

$\underbrace{\text{2 generated rings}}_{\substack{\text{all ideals are min} \\ \text{gen by } \leq 2 \text{ elements}}}$ are $\underbrace{\text{stable rings}}_{\substack{\text{all ideals are} \\ \text{stable.}}}$

Note: • Arf rings behave well under localization and completions

• Arf $\Rightarrow$ min mult $\Rightarrow$ Golod

extremal growth in the Betti numbers of the residue field

Trace Ideals & Integral Closures.

The set of trace ideals is closed under several closure operations.

Thm (Dao, –)

If $I \subseteq R$ is a trace ideal then so is:

- $\operatorname{Ann} \operatorname{Ann}(I)$ (L–, Pande)
- $\sqrt{I}$
- $\overline{I}$
- (R domain) $I^{**} = (I^{-1})^{-1}$ (Barucci 1984)

Trace Ideals & Stability

Prop (Dao, –) R local 1-dim Cohen Macaulay
I a reg trace ideal, TFAE:

(1) $I^2 = xI$, some nzd $x \in I$
(2) $I^2 \subseteq x$; "
(3) $I = (x) : I$; "
(4) $I \cong I^*$
(5) $I \cong \mathrm{End}_R(I)$
(6) $I \cong I^2$.

Lemma (Dao, _) I regular trace ideal
w/ principal reduction
& $\overline{I}$ stable, then $I = \overline{I}$

Cor R Arf ring s.t. any regular ideal has a principal reduction

$$\left\{ \begin{array}{l} \text{regular trace} \\ \text{ideals} \end{array} \right\} = \left\{ \begin{array}{l} \text{Integrally closed ideals} \\ \text{that contain the conductor} \end{array} \right\}.$$

(see also Isobe, Kobayashi).

Thm (Dao, –) R 1-dim local Cohen Macaulay ring s.t. every regular ideal has a principal reduction. TFAE

1) every regular trace ideal is stable
2) R is Arf.

Thm (Dao, –)

If $I = \tau_M(R)$ is regular & stable

then $I = \tau_{M^*}(R)$ & $I \in \operatorname{add} M^*$

i.e. I is a summand of a direct sum of copies of M^*

Idea: $\underbrace{Z(\mathrm{End}(M^*))}_{S} \overset{\text{Thm}}{=} \mathrm{End}(I) \overset{\text{trace}}{=} I^*$

M^* gens $M \otimes M^*$ gens $\tau_M(R) = I \overset{\text{stable}}{\cong} S$

$\Rightarrow \exists\ \Lambda$ s.t. $(M^*)^{\Lambda} \xrightarrow[S \text{ linear}]{} S \cong I$

$\Rightarrow I \cong S \in \operatorname{add} M^*$

Cor (Dao, —) R complete Arf domain where any regular ideal has a principal reduction

Then every reflexive module is isomorphic to a direct sum of integrally closed ideals

Cor In a complete Arf domain there are only finitely many classes of indecomposable reflexive modules.

(see also Dao, Maitra, Sridhar).

Thank You For Your

Attention!