Densely Computable Structures and Isomorphisms, Part I

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Generically and coarsely computable structures

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Generic case complexity

- Complexity results in computable structure theory often depend on the behavior of the hardest instances of the problem.

- For problems on groups, Kapovich, Myasnikov, Schupp, and Shpilrain (2003) proposed using the notion of asymptotic density to see whether a partial computable function could solve "almost all" instances of a problem.

- They showed that for a large class of finitely generated groups the classical decision problems, such as the word problem or the conjugacy problem, have linear time generic case complexity.

- Kapovich, Myasnikov, Schupp, and Shpilrain established that a finitely presented group with undecidable word problem, given by W. Boone, has a generically computable copy.

- Jockusch and Schupp (2012) introduced this topic to computability theory. They defined and investigated generically computable and coarsely computable sets of natural numbers.

- For $A \subseteq \omega$ and $n \geq 1$, the density of a set A up to n, denoted by $\rho_n(A)$, is
$$\frac{|A \cap \{0, 1, 2, \ldots, n-1\}|}{n}.$$

- The (*asymptotic*) *density* of A is $\rho(A) = \lim_{n \to \infty} \rho_n(A)$.

- For example, $A = \{2^n : n \in \omega\}$ has density 0.

- A is (*asymptotically*) *dense* if $\rho(A) = 1$.

- The *upper density* of A is $\limsup_{n \to \infty} \frac{|A \cap \{0,1,2,\ldots,n-1\}|}{n}$.

- As usual, computably enumerable is abbreviated by c.e.

- If A is a c.e. set with upper density 1, then A has a computable subset with upper density 1.

Generically and coarsely computable sets (Jockusch and Schupp)

- For $S \subseteq \omega$, let c_S denote the characteristic function of S.

- S is *generically computable* if there is a partial computable function $\varphi : \omega \to \{0, 1\}$ such that: $dom(\varphi)$ has asymptotic density 1, and $c_S \restriction dom(\varphi) = \varphi$.

- S is *coarsely computable* if there is a *total* computable function $\tau : \omega \to \{0, 1\}$ such that $\{x : c_S(x) = \tau(x)\}$ has asymptotic density 1.

  Equivalently, S is coarsely computable if there is a computable set T such that $S \triangle T$ has asymptotic density 0.

- (Jockusch and Schupp)

  There is a coarsely computable c.e. set that is not generically computable.

  There is a generically computable c.e. set that is not coarsely computable.

- A structure $\mathcal{C}$ for a finite language is *computable* if its domain C is computable and each relation of $\mathcal{C}$ is computable and each function of $\mathcal{C}$ is computable.

- A structure $\mathcal{D}$ for a finite language is *c.e.* if its domain D is c.e. and each relation of $\mathcal{D}$ is c.e. and each function of $\mathcal{D}$ is the restriction of a partial computable function to D.

Asymptotic density in $\omega \times \omega$

- Let $A \subseteq \omega$. Then A has asymptotic density δ in ω if and only if $A \times A$ has asymptotic density δ^2 in $\omega \times \omega$.

  Hence: A is (asymptotically) dense in ω iff $A \times A$ is (asymptotically) dense in $\omega \times \omega$.

- There is a computable dense set $R \subseteq \omega \times \omega$ such that for any infinite c.e. set $X \subseteq \omega$, the product $X \times X$ is not a subset of R.

Generically computable structures

- Consider a structure $\mathcal{A}$ for finite language with universe ω,

  with functions $\{f_i : i \in I\}$, each f_i of arity p_i, and

  relations $\{R_j : j \in J\}$, each R_j of arity r_j.

- We call $\mathcal{A}$ *generically computable* if $\mathcal{A}$ has a substructure $\mathcal{D}$ with a dense c.e. domain D, and there are partial computable functions $\{\phi_i : i \in I\}$ and $\{\psi_j : j \in J\}$ such that

  each ϕ_i agrees with f_i on D^{p_i} and

  each ψ_j agrees with c_{R_j} on the set D^{r_j}.

Example

- Let $\mathcal{M} = (\omega, A)$, where A is a unary relation.

- Assume that A is a generically computable set. Let a partial computable function φ be such that: $dom(\varphi)$ has density 1, and for every $x \in dom(\varphi)$, we have $c_A(x) = \varphi(x)$.

  Let $D = dom(\varphi)$. D is a c.e. set.

  Consider the substructure $\mathcal{D} = (D,\ A \cap D)$. Since D is c.e. and φ agrees with c_A on D, the structure $\mathcal{M}$ is generically computable.

- Conversely, if $\mathcal{M}$ is a generically computable structure, then A is a generically computable set.

Σ_n generically c.e. structures

- A substructure $\mathcal{B}$ is a Σ_n *elementary substructure* of $\mathcal{A}$ if for any infinitary Σ_n formula $\theta(x_1, \ldots, x_n)$ and $b_1, \ldots, b_n \in \mathcal{B}$:

  $\mathcal{A} \models \theta(b_1, \ldots, b_n)$ iff $\mathcal{B} \models \theta(b_1, \ldots, b_n)$

- A structure $\mathcal{A}$ is Σ_n *generically c.e.* if $\mathcal{A}$ has a c.e. substructure $\mathcal{D}$ with a dense (c.e.) domain D, such that $\mathcal{D}$ is also a Σ_n elementary substructure of $\mathcal{A}$.

- Clearly, a Σ_{n+1} generically c.e. structure is Σ_n generically c.e.

- Any c.e. structure is Σ_n generically c.e. for any n.

Generically computable and Σ_1 generically c.e. injection structures

- An *injection structure* $\mathcal{A} = (A, f)$ has a single unary function f that is $1 - 1$.

- Any c.e. injection structure is isomorphic to a computable injection structure.

- For $a \in A$, the *orbit* of a is

$$\mathcal{O}_f(a) = \{b \in A : (\exists n \in \omega)[f^n(a) = b \vee f^n(b) = a]\}$$

- Finite orbits are cycles, and infinite orbits can be of type ω or Z.

- The *character* of $\mathcal{A}$ is defined as:

  $\chi(\mathcal{A}) = \{\langle k, n\rangle : n, k > 0$ & there are $\geq n$ orbits of size $k\}$

- A c.e. injection structure $\mathcal{A}$ has a c.e. character $\chi(\mathcal{A})$.

- $K \subseteq \langle(\omega - \{0\}) \times (\omega - \{0\})\rangle$ is a *character* if for all $n > 0$ and k:

  $\langle k, n+1\rangle \in K \Rightarrow \langle k, n\rangle \in K$

- For any c.e. character K, there is a computable injection structure (A, f) with character K and any specified number of orbits of type ω and Z.

- An injection structure $\mathcal{A} = (\omega, f)$ has a *generically computable copy* iff at least one of the following two conditions hold:

  1. $\mathcal{A}$ has an infinite orbit;

  2. $\chi(\mathcal{A})$ has an infinite c.e. subset.

- $\mathcal{A} = (\omega, f)$ has a Σ_1 *generically c.e. copy* iff

  (i) $\mathcal{A}$ has a computable copy iff

  (ii) $\chi(\mathcal{A})$ is a c.e. set iff

  (iii) $\mathcal{A}$ has a Σ_2 generically c.e. copy.

Computable and c.e. equivalence structures

- Consider an equivalence structure $\mathcal{A} = (A, E)$.

- The *character* of $\mathcal{A}$ (or E) is defined as:

  $\chi(\mathcal{A}) = \{\langle k, n \rangle : n, k > 0$ & there are $\geq n$ equivalence classes of size $k\}$

- Bounded character: there is a finite bound on size k.

- If A and E are c.e., then the character $\chi(\mathcal{A})$ is a Σ^0_2 set.

- (Calvert, Cenzer, Harizanov, Morozov)

  For any Σ^0_2 character K, there is a computable equivalence structure $\mathcal{A}$ with character K and infinitely many infinite equivalence classes.

- (Cenzer, Harizanov, Remmel)

  For any Σ^0_2 character K, there is a c.e. equivalence structure, even with a computable domain, with character K and with any finite number $r \geq 1$ of infinite equivalence classes.

Generically computable equivalence structures

A surprising result:

- Every equivalence structure $\mathcal{A} = (\omega, E)$ has a generically computable copy.

Σ_1 and Σ_2 generically c.e. equivalence structures

- A function $h : \omega^2 \to \omega$ is a Khisamiev s_1-*function* if for all i, t,

  $h(i,t) \leq h(i,t+1)$,

  $m_i = \lim_{t \to \infty} h(i,t)$ exists, and

  $m_0 < m_1 < \cdots < m_i < \cdots$

- Let $\mathcal{A} = (A, E)$ be a c.e. equivalence structure with no infinite equivalence classes and an unbounded character.

  Then there is a computable s_1-function h such that $\mathcal{A}$ contains an equivalence class of size m_i for each $i \in \omega$.

- We say that a character K has an s_1-function h if $\langle m_i, 1\rangle \in K$ for each i.

- For every Σ^0_2 character K that is either bounded or has a computable s_1-function, there is a computable equivalence structure $\mathcal{A}$ with character K and no infinite equivalence classes.

- If $\mathcal{A}$ is c.e. equivalence structure with no infinite equivalence classes, then $\mathcal{A}$ is isomorphic to a computable structure.

- An equivalence structure $\mathcal{A} = (\omega, E)$ has a Σ_1 *generically c.e. copy* iff at least one of the following conditions hold:

  1. $\chi(\mathcal{A})$ is bounded;
  2. $\chi(\mathcal{A})$ has a Σ_2^0 subcharacter K with a computable s_1-function;
  3. $\chi(\mathcal{A})$ has a Σ_2^0 subcharacter H, and $\mathcal{A}$ has an infinite class;
  4. $\mathcal{A}$ has infinitely many infinite classes.

- $\mathcal{A} = (\omega, E)$ has a Σ_2 *generically c.e. copy* iff

  (i) $\mathcal{A}$ has a c.e. copy iff

  (ii) $\mathcal{A}$ has a Σ_3 generically c.e. copy.

Coarsely computable structures

- A structure $\mathcal{A}$ is *coarsely computable* if there are a computable structure $\mathcal{E}$ and a dense set D, which is the domain of a structure $\mathcal{D}$ that is a substructure of both $\mathcal{A}$ and of $\mathcal{E}$ (all relations and functions agree on D):

$$\mathcal{D} \subseteq \begin{matrix} \mathcal{A} \\ \mathcal{E} \end{matrix}$$

- $\mathcal{M} = (\omega, A)$ is a coarsely computable structure iff A is a coarsely computable set.

  There is a generically computable structure that is not coarsely computable, and there is a coarsely computable structure that is not generically computable.

Σ_n coarsely c.e. structures

- A structure $\mathcal{A}$ is Σ_n *coarsely c.e.* if there are a c.e. structure $\mathcal{E}$ and a dense set D, which is the domain of a substructure $\mathcal{D}$ that is a Σ_n elementary substructure of both $\mathcal{A}$ and $\mathcal{E}$ (all relations and functions agree on D):

$$\mathcal{D} \preceq_n \begin{matrix} \mathcal{A} \\ \mathcal{E} \end{matrix}$$

- Clearly, a Σ_{n+1} coarsely c.e. structure is Σ_n coarsely c.e.

- A Σ_0 coarsely c.e. structure is also called a *coarsely c.e.* structure.

  Every coarsely computable structure is a coarsely c.e. structure.

Coarse computability for injection structures

Generic computability vs coarse computability:

- There is a generically computable injection structure that is not coarsely computable.
- Every generically computable injection structure has a coarsely computable copy.

- There is a coarsely computable injection structure with no generically computable copy.

- *Proof idea*. Let $S \subseteq \omega - \{0\}$ be a dense *immune* set (does not contain an infinite c.e. subset).

  Build a coarsely computable injection structure $\mathcal{A}$ with character

  $\{\langle k, i\rangle : k \in S \wedge i \in \{1, 2\}\}$ and no infinite orbits such that

  if $\mathcal{B}$ were a generically computable copy of $\mathcal{A}$, then $\chi(\mathcal{B}) = \chi(\mathcal{A})$ would contain an infinite c.e. subset C.

  Then $\{k : \langle k, 1\rangle \in C \vee \langle k, 2\rangle \in C\}$ would be an infinite c.e. subset of S, a contradiction.

- There is an injection structure that does not have a coarsely computable copy.

- *Proof idea*. Build an infinite set $S \subseteq \omega$ such that an injection structure $\mathcal{A}$ with character $\chi(\mathcal{A}) \subseteq \{\langle k, 1 \rangle : k \in S\}$ cannot be coarsely computable.

- *Question*: Characterize injection structures that have coarsely computable copies.

- An injection structure $\mathcal{A} = (\omega, f)$ has a Σ_1 *coarsely c.e. copy* iff

  (i) $\mathcal{A}$ has a computable copy iff

  (ii) $\chi(\mathcal{A})$ is a c.e. set.

Coarse computability for equivalence structures

- Recall: Every equivalence structure has a generically computable copy.

- There is a Σ_1 coarsely c.e. equivalence structure with no Σ_1 generically c.e. copy.

- There is an equivalence structure with no Σ_1 coarsely c.e. copy.

- *Question*: Characterize equivalence structures that have Σ_1 coarsely c.e. copies.

- Let $\mathcal{A}$ be an equivalence structure with an infinite class, or with a bounded character, or with an unbounded character that has a computable s_1-function.

  Then $\mathcal{A}$ has a Σ_2 *coarsely c.e. copy* iff

  (i) $\chi(\mathcal{A})$ is a Σ_2^0 set iff

  (ii) $\mathcal{A}$ has a c.e. copy.

- Let $\mathcal{A}$ be an equivalence structure with no infinite classes, with an unbounded character with no computable s_1-function.

  Then $\mathcal{A}$ has a Σ_2 *coarsely c.e. copy* iff

  $\chi(\mathcal{A})$ is a Σ_2^0 set, and for some finite k, $\mathcal{A}$ has infinitely many classes of size k.

- For any equivalence structure $\mathcal{A}$,

  $\mathcal{A}$ has a Σ_3 *coarsely c.e.* copy iff

  $\mathcal{A}$ has a c.e. copy.

THANK YOU!