Densely Computable Structures and Isomorphisms, Part II: Math is About Functions

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July 20, 2022

Theorem

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2. *There is a computable algebraic field K such that there exist two algebraic closures J_1, J_2 of K such that there is no computable isomorphism $J_1 \to J_2$.*

Definition

A computable structure $\mathcal{A}$ is said to be computably categorical if and only if any two copies $\mathcal{A}_0 \cong \mathcal{A}_1 \cong \mathcal{A}$ are isomorphic via a computable isomorphism.

Question

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Remark

Composition does not always play nicely with exceptions.

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1. Maybe the only "densely" computable isomorphisms are the ones where there's actually a computable isomorphism.
2. Maybe there are "densely" computable isomorphisms everywhere.

Definition

Let $F : \mathbb{N} \to \mathbb{N}$ be a total function. We say that F is generically computable if there is a partial computable function θ such that $\theta = F$ on the domain of θ, and such that the domain of θ has asymptotic density 1.

Definition

1. We say that an isomorphism $F : \mathcal{A} \to \mathcal{B}$ from a structure $\mathcal{A}$ to a structure $\mathcal{B}$ is a generically computable isomorphism if there are a c.e. set C of asymptotic density one and a partial computable function θ with $C \subseteq Dom(\theta)$, which satisfy the following:

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   1. C is the domain of a substructure $\mathcal{C}$ of $\mathcal{A}$;
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   2. $F(x) = \theta(x)$ for all $x \in C$;
   3. The image $F[C]$ has asymptotic density one and is the domain of a substructure of $\mathcal{B}$.
2. A structure $\mathcal{A}$ is generically computably isomorphic to a structure $\mathcal{B}$ if there is a generically computable isomorphism F mapping $\mathcal{A}$ to $\mathcal{B}$.

Example

Let A, B be dense coinfinite c.e. sets. Then the structures $(\mathbb{N}, A)$ and $(\mathbb{N}, B)$ are generically computably isomorphic.

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To do this, take one-to-one enumerations of the two sets, and let θ be the function matching corresponding elements. We define F_0 on A to be θ, and extend F_0 arbitrarily to a total bijection.

If $(\mathbb{N}, A)$ and $(\mathbb{N}, B)$ were computably isomorphic, then A and B would be 1-equivalent.

Theorem

Let $\mathcal{A}$ be an equivalence structure. Then $\mathcal{A}$ is computably categorical if and only if one of the following holds:

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- *$\mathcal{A}$ has only finitely many finite classes, or*
- *$\mathcal{A}$ has finitely many infinite classes, there is a bound on the size of the finite classes, and there is at most one k such that $\mathcal{A}$ has infinitely many classes of size k.*

Example

A $(1, 2)$-equivalence structure is an equivalence structure having

- infinitely many classes of size 1,
- infinitely many classes of size 2,
- and nothing else.

Definition

We say that the equivalence structure $\mathcal{A} = (\mathbb{N}, E)$ has generic character K for a finite subset K of $\mathbb{N} - \{0\}$ if, for each $k \in K$, the set $\mathcal{A}(k)$ has positive asymptotic density and the union $\bigcup_{k \in K} \mathcal{A}(k)$ has asymptotic density one.

Theorem

1. *If $\mathcal{A}$ and $\mathcal{B}$ are computable $(1, 2)$-equivalence structures, each having generic character $\{2\}$, then $\mathcal{A}$ and $\mathcal{B}$ are generically computably isomorphic.*

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2. *Any computable $(1,2)$-equivalence structure $\mathcal{A}$ with generic character $\{2\}$ is generically computably isomorphic to a computable structure $\mathcal{C}$ in which the set of elements of size 2 is computable.*

Theorem

If $\mathcal{A}$ and $\mathcal{B}$ are generically c.e. $(1,2)$-equivalence structures, each having generic character $\{2\}$, then $\mathcal{A}$ and $\mathcal{B}$ are generically computably isomorphic.

Theorem

If $\mathcal{A}$ and $\mathcal{B}$ are isomorphic computable equivalence structures with finitely many infinite classes such that the infinite classes constitute a set of asymptotic density 1 in each structure, then $\mathcal{A}$ and $\mathcal{B}$ are generically computably isomorphic.

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It follows that any such structure $\mathcal{A}$ is generically computably isomorphic to a computable structure $\mathcal{C}$ in which the set of elements that belong to infinite classes is computable.

Theorem

Let $\mathcal{A}$ and $\mathcal{B}$ be isomorphic generically c.e. equivalence structures, each with a single infinite class of asymptotic density 1. Then $\mathcal{A}$ and $\mathcal{B}$ are generically computably isomorphic.

Definition

We say that structures $\mathcal{A}$ and $\mathcal{B}$ are weakly generically computably isomorphic if there are a c.e. set C of asymptotic density one, a bijection $F : \mathcal{A} \to \mathcal{B}$, and a partial computable function θ with $C \subseteq Dom(\theta)$, which satisfy the following:

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(i) C is the domain of a substructure $\mathcal{C}$ of $\mathcal{A}$;

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(ii) $F(x) = \theta(x)$ for all $x \in C$;

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(i) C is the domain of a substructure $\mathcal{C}$ of $\mathcal{A}$;

(ii) $F(x) = \theta(x)$ for all $x \in C$;

(iii) $F[C]$ has asymptotic density one and is the domain of a substructure $\mathcal{C}_1$ of $\mathcal{B}$.

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We say that structures $\mathcal{A}$ and $\mathcal{B}$ are weakly generically computably isomorphic if there are a c.e. set C of asymptotic density one, a bijection $F : \mathcal{A} \to \mathcal{B}$, and a partial computable function θ with $C \subseteq Dom(\theta)$, which satisfy the following:

(i) C is the domain of a substructure $\mathcal{C}$ of $\mathcal{A}$;

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(iii) $F[C]$ has asymptotic density one and is the domain of a substructure $\mathcal{C}_1$ of $\mathcal{B}$.

(iv) θ is an isomorphism from $\mathcal{C}$ to $\mathcal{C}_1$.

Theorem

For any $k > 1$ *and any rational number* p *with* $0 < p \leq 1$, *there exist computable* $(1, k)$*-equivalence structures* $\mathcal{A}$ *and* $\mathcal{B}$ *such that* $\mathcal{A}(1)$ *and* $\mathcal{B}(1)$ *each have asymptotic density* p, *which are not weakly generically computably isomorphic.*

Remark

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We can at least show that this is symmetric and reflexive.

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Proposition

Density-preserving generically computable and weakly generically computable isomorphism are transitive.

Question

We know that the composition of two generically computable isomorphisms need not be a generically computable isomorphism. Is it transitive anyway?

Proposition

Suppose that a structure $\mathcal{A}$ has a c.e. substructure $\mathcal{D}$ on a dense computable set, the domain of $\mathcal{D}$. Then there are a c.e. structure $\mathcal{B}$ and a weakly generically computable density preserving isomorphism from $\mathcal{A}$ to $\mathcal{B}$.

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Suppose that a structure $\mathcal{A}$ has a c.e. substructure $\mathcal{D}$ on a dense computable set, the domain of $\mathcal{D}$. Then there are a c.e. structure $\mathcal{B}$ and a weakly generically computable density preserving isomorphism from $\mathcal{A}$ to $\mathcal{B}$.

Moreover, if there is a weakly generically computable density preserving isomorphism from $\mathcal{A}$ to a c.e. structure, then $\mathcal{A}$ is generically computable.

Definition

A function F is said to be coarsely computable if and only if there is a total computable function θ such that $\{n : F(n) = \theta(n)\}$ has asymptotic density 1.

Definition

We say that an isomorphism $F : \mathcal{A} \to \mathcal{B}$ from a structure $\mathcal{A}$ to a structure $\mathcal{B}$ is a coarsely computable isomorphism if there are a set C of asymptotic density one and a (total) computable function θ such that:

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1. C is the domain of a substructure $\mathcal{C}$ of $\mathcal{A}$;
2. $F(x) = \theta(x)$ for all $x \in C$;
3. The image $F[C]$ has asymptotic density one and is the domain of a substructure of $\mathcal{B}$.

Theorem

Let $\mathcal{A} = (A, R)$ and $\mathcal{B} = (B, S)$ be isomorphic equivalence structures with generic character $\{1\}$ (that is, both $\mathcal{A}(1)$ and $\mathcal{B}(1)$ have asymptotic density one). Then there is a density preserving coarsely computable isomorphism between $\mathcal{A}$ and $\mathcal{B}$.

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Theorem

Let $\mathcal{A}$ have generic character $\{1\}$, and let $\mathcal{B}$ be an equivalence structure, with a coarsely computable isomorphism $F : \mathcal{A} \to \mathcal{B}$. Then $\mathcal{B}(1)$ has asymptotic density 1.

Definition

We say that structures $\mathcal{A}$and $\mathcal{B}$ are weakly coarsely computably isomorphic if there is a set C of asymptotic density one, a bijection $F : \mathcal{A} \to \mathcal{B}$ and a total computable function θ, which satisfy the following:

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1. $\theta(x) = F(x)$ for all $x \in C$;
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3. The image $F[C]$ also has asymptotic density one and is the universe of a substructure of $\mathcal{B}$.

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3. The image $F[C]$ also has asymptotic density one and is the universe of a substructure of $\mathcal{B}$.
4. θ is an isomorphism from $\mathcal{C}$ to its image.

Proposition

If a structure $\mathcal{A}$ is coarsely computable, then there is a density preserving weakly coarsely computable isomorphism from $\mathcal{A}$ to a computable structure.

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If a structure $\mathcal{A}$ is coarsely computable, then there is a density preserving weakly coarsely computable isomorphism from $\mathcal{A}$ to a computable structure.

Moreover, if there is a weakly coarsely computable isomorphism from $\mathcal{A}$ to a computable structure, then $\mathcal{A}$ is coarsely computable.

Proof: Let $\mathcal{E}$ be a computable structure, and let D be a dense set such that the structure $\mathcal{D}$ with domain D is a substructure of both $\mathcal{A}$ and $\mathcal{E}$, and all relations and functions agree on D.

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Then the identity function serves as the desired isomorphism.

The converse is immediate from the definitions. $\square$

Theorem

Suppose that $\mathcal{A} = (\mathbb{N}, R)$ *is a computable* $(1, 2)$*-structure such that* $\mathcal{A}(1)$ *has asymptotic density q, where* $0 < q < 1$*. Then there are a computable structure* $\mathcal{B}$*, such that* $\mathcal{B}(1)$ *is a computable set with computable asymptotic density q, and a density preserving weakly coarsely computable isomorphism from* $\mathcal{A}$ *to* $\mathcal{B}$*.*

Theorem

Suppose that $\mathcal{A}$ and $\mathcal{B}$ are computable $(1,2)$-equivalence structures with domain $\mathbb{N}$ such that the asymptotic density of $\mathcal{A}(1)$ and $\mathcal{B}(1)$ both equal the same computable real q. Then $\mathcal{A}$ and $\mathcal{B}$ are weakly coarsely computably isomorphic.

Conjecture

Let $K = \{k_1, \ldots, k_n\} \subseteq \mathbb{N} - \{0\}$ be a finite set and let $q_1, \ldots, q_n$ be positive reals such that $q_1 + \cdots + q_n = 1$. Let $\mathcal{A}$ and $\mathcal{B}$ be computable equivalence structures such that $\mathcal{A}(k_i)$ and $\mathcal{B}(k_i)$ have asymptotic density q_i for each i. Then $\mathcal{A}$ and $\mathcal{B}$ are weakly coarsely computably isomorphic.

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