# Densely Computable Structures and Isomorphisms Current and Future Research

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# Outline

Three areas for research on densely computable structures

(1) Effectively closed sets

(2) Models of Peano Arithmetic

(3) Abelian groups

# Members of Effectively Closed Sets

It is well known that there are nonempty $\Pi^0_1$ classes with no computable members

### Theorem

*There is an effectively closed set $Q \subset \{0,1\}^\omega$ with no generically computable members.*

The proof is via a strengthening of the construction of the Diagonally Non-Computable functions

# Proof

*Requirement* $R_e$: For $X \in Q$, $\{i : X(i) = \phi_e(i)\}$ is not dense

Partition $\omega \setminus \{0\}$ into a computable sequence of sets

$$A_e = \{2^e(2n+1) : n \in \omega\}$$

Then $A_e$ has asymptotic density $2^{-e-1}$ for each $e$.

Define the effectively closed set $Q$ as follows:

$$X \in Q \iff (\forall e)(\forall i \in A_e)[\phi_e(i)\downarrow \Rightarrow X(i) \neq \phi_e(i)]$$

$Q$ is nonempty (in fact perfect)

Now let $X \in Q$ and consider $D = \{i : X(i) = \phi_e(i)\}$

$D$ is disjoint from $A_e$ and so has density $\leq 1 - 2^{-e-1}$

Thus $D$ is not asymptotically dense

It follows that $Q$ has no generically computable members.

# Questions

## Conjecture

*There is a $\Pi^0_1$ class of positive Lebesgue measure with no densely computable member*

Notions of computability at density $r$ have also been studied.

$X$ is *partial computable at density $r$* if there is a partial computable $\phi$ so $\{n : X(n) = \phi(n)\}$ has density at least $r$

Then one can define $\alpha(X)$ to be the supremum of the $r$ such that $X$ is partial computable at density at least $r$.

## Question

1. *Does every $\Pi^0_1$ $Q$ contain an element $X$ with $\alpha(X) = 1$?*
2. *For any $\epsilon > 0$ and any $\Pi^0_1$ $Q$, does $Q$ have an element which is partial computable at density $1 - \epsilon$*

# Representations

The set of complete consistent extensions of a computable (propositional) theory may be represented as a $\Pi^0_1$ class

## Question

1. *Is there a good notion of a generically computable theory?*
2. *What can be said about the set of extensions of a generically computable theory?*

Similar questions can be posed for generically computable graphs and their colorings and many other problems

# Models of Peano Arithmetic

Tennenbaum's Theorem: There is no computable non-standard model of Peano arithmetic

Using the methods shown earlier, we can prove:

## Theorem

*There is a generically computable non-standard model of Peano Arithmetic;*

*In fact, every countable non-standard model of Peano Arithmetic has a generically computable copy*

# The standard non-standard model

In the usual non-standard model $\mathcal{M}$, the standard part $\mathbb{N}$ is an elementary submodel

The generically computable copy will then be $\Sigma_n$ generically c.e. for all $n$.

## Question

*Is there a $\Sigma_1$-generically computable non-standard model of Peano Arithmetic where the dense computable substructure has a non-standard component*

# Isomorphisms

### Theorem

*Let $\mathcal{M}_1$ and $\mathcal{M}_2$ be generically computable non-standard models of Peano Arithmetic in which the standard parts of each are the necessary dense computable substructures.*

*Then $\mathcal{M}_1$ and $\mathcal{M}_2$ are generically computably isomorphic*

# Countable Abelian Groups

## Definition

*Let $\mathcal{A}$ be an Abelian group and let $p$ be a prime number.*

1. *$\mathcal{A}$ is a $p$-group if every element has order a power of $p$.*
2. *$\mathcal{A}[p]$ is the subgroup of elements with order a power of $p$.*
3. *The $p$-height $ht_p^{\mathcal{A}}(x)$ of an element $x \in \mathcal{A}$ is the largest $n$ such that $p^n | x$, that is, there exists $y$ such that $p^n y = x$.*
4. *A subgroup $\mathcal{B}$ of $\mathcal{A}$ is pure if, for every prime $q$ and every $b \in B$, $ht_q^{\mathcal{B}}(b) = ht_q^{\mathcal{A}}(b)$. The subscript $q$ will be omitted if it is clear from the context.*
5. *$\mathcal{A}$ is divisible if every element of $\mathcal{A}$ has infinite height, that is, for every $x \in \mathcal{A}$ and every $n \in \mathbb{N}$, there exists $y \in \mathcal{A}$ such that $x = n \cdot y$.*
6. *A group is reduced if it has no divisible subgroup.*

# Background [Kaplansky, Fuchs]

## Theorem (Baer)

*Every Abelian group is a direct sum of a divisible group and a reduced group.*

## Theorem

*Any torsion group is the direct sum of p-groups $A[p]$.*

## Theorem (Prüfer)

*A countable Abelian p-group is a direct sum of cyclic groups if and only if it contains no elements of infinite height.*

## Theorem (Szele)

*Let $\mathcal{B}$ be a subgroup of the Abelian p-group $\mathcal{A}$ such that $\mathcal{B}$ is the direct sum of cyclic subgroups of the same order $p^k$, for some finite $k$. Then $\mathcal{B}$ is a direct summand of $\mathcal{A}$ if and only if $\mathcal{B}$ is a p-pure subgroup of $\mathcal{A}$.*

# Character

## Corollary

*Suppose the countable Abelian p-group $\mathcal{A} = \mathcal{C} \oplus \mathcal{D}$, where $\mathcal{C}$ has no elements of infinite height and $\mathcal{D}$ is divisible. Then $\mathcal{A}$ has the form $\bigoplus_{i<\omega} \mathbb{Z}(p^{n_i}) \oplus \bigoplus_{i \leq k} \mathbb{Z}(p^\infty)$, where $k \leq \omega$.*

For such $p$-groups, the *character* $\chi(\mathcal{A})$ of $\mathcal{A}$ is

$$\{(k, n) \in (\omega \backslash \{0\})^2 : \mathcal{A} \text{ has at least } n \text{ factors of the form } \mathbb{Z}(p^k)\}$$

Computable Abelian $p$-groups were studied by A. Morozov and the authors

See Khisamiev [Handbook of Recursive Mathematics] for more

# Complexity of the Character

## Proposition (Kulikov)

*For any countable Abelian p-group $\mathcal{A}$ and any $n, k \geq 1$, $(n, k) \in \chi(\mathcal{A})$ if and only if $\mathcal{A}$ has a pure subgroup isomorphic to $\bigoplus_{i<n} \mathbb{Z}(p^k)$.*

## Theorem

*[Khisamiev] For any computable p-group $\mathcal{A}$, $\chi(\mathcal{A})$ is a $\Sigma_2^0$ set.*

## Proposition

*Let $K$ be a $\Sigma_2^0$ character. Then there is a computable equivalence structure $\mathcal{A} = (\omega, E)$ with character $K$ and with infinitely many infinite equivalence classes.*

# Generically Computable Copies

## Proposition

*Every countable Abelian p-group $\mathcal{A}$ has a generically computable copy.*

*Proof Sketch:* The proof is in two steps

First, show that $\mathcal{A} = (\omega, +_A)$ has a subgroup $\mathcal{B}$ which is somorphic to a computable group

Second, obtain a computable group $\mathcal{D} = (D, +_D)$ isomorphic to $B$ with universe $D$ a dense co-infinite set, and then extend $\mathcal{D}$ to generically computable $\mathcal{C} = (\omega, +_C)$ isomorphic to $\mathcal{A}$.

The second step is similar to that for equivalence structures

# Three Cases

The first step is in three cases:

1. Every element of $\mathcal{A}$ has finite height
   Then $\mathcal{A}$ will have a subroup isomorphic to $\oplus_{i<\omega}\mathbb{Z}(p)$
   This is because every $\mathbb{Z}(p^k)$ has a subgroup of type $\mathbb{Z}(p)$
2. $\mathcal{A}$ has a divisible subgroup $B$
3. $\mathcal{A}$ has some element of infinite height
   Then there exists $b$ such that $\{x : px = b\}$ is infinite
   It will follows that $\mathcal{A}$ has a subgroup of bounded order

# Countable Abelian non-$p$-Groups

For any countable Abelian group $\mathcal{A}$, let

$$\mathcal{A}[p] = \{x \in \mathcal{A} : p^n x = 0 \text{ for some } n\}$$

## Theorem

*A countably infinite Abelian group has a generically computable copy if and only if either*

1. $\mathcal{A}[p]$ *is infinite for some prime* $p$*, or*
2. $\{p : \mathcal{A}[p] \neq 0\}$ *has an infinite c.e. subset.*

# Proof Sketch

First let $\mathcal{C}$ be a generically computable copy and $\mathcal{D} = (D, +_D)$ a c.e. subgroup of $\mathcal{C}$

If $\mathcal{D}[p]$ is finite for all $p$, then $\mathcal{D}[p] \neq 0$ for infinitely many $p$

So $\{p : \mathcal{D}[p] \neq 0\}$ is an infinite c.e. subset of $\{p : \mathcal{C}[p] \neq 0\} = \{p : \mathcal{A}[p] \neq 0\}$

For the other direction, first let $\mathcal{A}[p]$ be infinite

Then $\mathcal{A}[p]$ has a generically computable copy $\mathcal{B}$

Second, let $P$ be an infinite c.e. set of primes with $\mathcal{A}[p] \neq 0$

Then $\mathcal{A}$ will have a subgroup isomorphic to $\bigoplus_{p \in P} \mathbb{Z}(p)$

# Generically Computable Isomorphisms

Previous work showed that the Abelian $p$-group $\bigoplus_{i<\omega} \mathbb{Z}(p) \oplus \bigoplus_{i<\omega} \mathbb{Z}(p^2)$ is not computably categorical

That is, there are computable copies which are not computably isomorphic

## Theorem

*Let $\mathcal{A}$ and $\mathcal{B}$ be computable Abelian $p$-groups each isomorphic to $\bigoplus_{i<\omega} \mathbb{Z}(p) \oplus \bigoplus_{i<\omega} \mathbb{Z}(p^2)$ such that the elements of order $p^2$ are asymptotically dense*

*Then $\mathcal{A}$ and $\mathcal{B}$ are generically computably isomorphic*

# Proof Sketch

The sets of elements of order $p^2$ are computable in $\mathcal{A}$ and $\mathcal{B}$

Thus we can construct sequences $a_0, a_1, \ldots$ and $b_0, b_1, \ldots$ of independent elements of order $p^2$ from each structure along with a (partial) computable isomorphism $\phi$ mapping $\mathcal{A}_2 =< a_0 > \oplus < a_1 > \cdots \oplus \ldots$ to $\mathcal{B}_2 =< b_0 > \oplus < b_1 > \cdots \oplus \ldots$

The subgroups $\mathcal{A}_2$ and $\mathcal{B}_2$ are pure, so by Szele's Theorem there exist $\mathcal{A}_1$ and $\mathcal{B}_1$ of type $\bigoplus_{i<\omega} \mathbb{Z}(p)$ such that $\mathcal{A} = \mathcal{A}_1 \oplus \mathcal{A}_2$ and $\mathcal{B} = \mathcal{B}_1 \oplus \mathcal{B}_2$.

Then we can extend $\phi$ to an isomorphism from $\mathcal{A}$ to $\mathcal{B}$

# Problems for Abelian Groups

## Conjecture

*Let $\mathcal{A}$ and $\mathcal{B}$ be computable Abelian p-groups each isomorphic to $\bigoplus_{i<\omega} \mathbb{Z}(p) \oplus \bigoplus_{i<\omega} \mathbb{Z}(p^2)$ such that the $\mathbb{Z}(p^2)$ factor of each has the same computable density q*

*Then $\mathcal{A}$ and $\mathcal{B}$ are coarsely computably isomorphic*

## Problem

*Characterize the $\Sigma_n$-generically computable Abelian groups*

# Other Structures

Generically and Coarsely Computable Structures and Isomorphism for other structures

Fields, Rings, Non-Abelian Groups, Orderings, Graphs, Trees, ...