# Strong minimality and algebraic relations between solutions for Poizat's family of equations

## Decidability, Definability and Computability in Number Theory

## MSRI

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Joint work with Jim Freitag, Remí Jaoui and Ronnie Nagloo

Work in $\mathbb{K}$ be a large differentially closed field.

A definable $X \subset \mathbb{K}^n$ is *strongly minimal* if every definable subset of $X$ is finite or cofinite.

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Shelah Philosophy–Finite rank sets are analyzed by successively analyzing strongly minimal sets.

Goal: Try to understand the strongly minimal subsets and the algebraic relationships between them.

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Strongly minimal sets $X$ and $Y$ are *non-orthogonal* $X \not\perp Y$ if there is a definable generically finite-to-finite $R \subset X \times Y$.

# Hrushovski–Sokolović Trichotomy

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*2) If $A$ is a simple abelian variety not isomorphic to a variety defined over $C$ and $A^\sharp$ is the closure in the Kolchin topology of $Tor(A)$, then $A^\sharp$ is a non-trivial locally modular strongly minimal set (We call $A^\sharp$ a* Manin kernel.*)*

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What can be said about trivial strongly minimal sets?

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• Hrushovski and Itai found strongly minimal examples on curves of genus$> 1$ defined over $C$.

# Higher order examples

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- Nagloo and Pillay showed that generic Painlevé equations are strongly minimal and trivial, for example

$$P_{II}(\alpha): \; X'' = 2X^3 + tX + \alpha$$

is trivial strongly minimal, where $\alpha \in C$ is transcendental and $t' = 1$. They also studied algebraic relations between solutions to different Painlevé equations.

• Freitag and Scanlon proved that the third order differential equation satisfied by the $j$-function is trivial strongly minimal. This was the first non $\aleph_0$-categorical example.

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• Jaoui has examples based on Riemannian folitations

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**Theorem (DeVilbis–Freitag)**

*If $f(X)$ is a generic differential polynomial of order $n > 1$ and degree $d \geq 2n + 4$, then $f(X) = 0$ is strongly minimal.*

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order 2 $\Rightarrow \perp C$;

defined over $C \Rightarrow \perp A^\sharp$ for any Manin kernel;

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**Theorem**

*$V_f$ is strongly minimal if and only there is no $g \in C(z)$ with $f = \frac{dg}{dz}$.*

$\Rightarrow$ If $f = \frac{dg}{dz}$ and $z' = g(z) + c$ for some $c \in C$, then $z'' = f(z)z'$.
Thus there is an infinite family of order 1 differential algebraic subvarieties of $V_f$.

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where $g(z) \in C(z)$, and $c_1, \ldots, c_n, \alpha_1, \ldots, \alpha_n \in C$.

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Suppose $f(z)$ has no an antiderivative.
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Consider the power series expansion

$$f(z) = \sum_{n=m}^{\infty} a_n z^m$$

then $a_{-1} \neq 0$. We call $a_{-1}$ the *residue* at 0.

If $V_f$ is not strongly minimal, then we can find a differential field $(K, \delta)$ and $z \in V_f$ transcendental over $K$ such that $z$ and $z'$ are algebraically dependent over $K$.

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Define a derivation $D$ on $K\langle\langle z \rangle\rangle$ such that

$$D\left(\sum a_i z^i\right) = \sum \delta(a_i) z^i + u \sum i a_i z^{i-1}$$

$D$ extends the natural derivation on $K(z, z')$.

Let $z' = u = \sum_{i=0}^{\infty} a_i z^{r+\frac{i}{n}}$, where $v(u) = r$.
Then

$$z'' = D(u) = \sum \delta(a_i) z^{r+\frac{i}{n}} + u \sum \left( r + \frac{i}{n} \right) a_i z^{r+\frac{i}{n}-1}$$

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The coefficient of $z^{-1}$ on the right summand is 0.
Thus we can not have $\frac{z''}{z'} = f(z)$, a contradiction.

# Liénard equations

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$$z'' + f(z)z' + g(z) = 0$$

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**Corollary**

*Suppose $f$ has no antiderivative in $C(z)$. Consider*

$$z'' + f(z)z' + \frac{\sum_{i=0}^{n} c_i z^i}{\sum_{j=0}^{m} d_j z^j} = 0$$

*where $c_0, \ldots, c_n, d_0, \ldots, d_m$ are algebraically independent constants.*

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Here $z'' + f(z)z' = 0$ is orthogonal to the constants, so the generic fiber is as well.

# Algebraic relations between solutions

Suppose $f, g \in C(z)$ have no antiderivatives in $C$ (possibly $f = g$).
Suppose $K \supseteq C$ is a differential field $x \in V_f$, $y \in V_g$ are each transcendental over $K$.

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Ingredients:

- basics on trivial strongly minimal sets;
- connections between Kähler differentials and transcendence, building on the work of Ax, Rosenlicht and Brestovski.

Suppose $x \in V_f$, $y \in V_g$ and $y$ is algebraic over $C(x)$.
$x'' = f(x)x'$ and $y'' = g(y)y'$.
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As $x$ satisfies no non-trivial first order differential equation we must have $\phi''(x) = 0$ and thus $\phi(x) = ax + b$.

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Since $\phi'(x) = a \neq 0$.
We must also have $f(x) = g(\phi(x))$.

## Lemma

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*ii) If* $x, y \in V_f$ *and* $y \in \mathrm{cl}(x)$*, there is* $\phi$ *as above with* $f = f \circ \phi$ *and* $\phi(x) = y$*.*

**Lemma**

*i) If $f \neq g$ and $V_f \not\perp V_g$, then $g = f \circ \phi$ for some affine transformation $\phi(x) = ax + b$;*
*ii) If $x, y \in V_f$ and $y \in \mathrm{cl}(x)$, there is $\phi$ as above with $f = f \circ \phi$ and $\phi(x) = y$.*

For most $f$ there in no nontrivial $\phi$ with $f = f \circ \phi$, so $\mathrm{cl}(x) = \{x\}$.
But it is possible

## Lemma

*i) If $f \neq g$ and $V_f \not\perp V_g$, then $g = f \circ \phi$ for some affine transformation $\phi(x) = ax + b$;*
*ii) If $x, y \in V_f$ and $y \in \mathrm{cl}(x)$, there is $\phi$ as above with $f = f \circ \phi$ and $\phi(x) = y$.*

For most $f$ there in no nontrivial $\phi$ with $f = f \circ \phi$, so $\mathrm{cl}(x) = \{x\}$.
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1) $f(x) = \frac{1}{x-a} - \frac{1}{x-b}$ and $\phi(x) = -x + a + b$;
2) $f(x) = \frac{1}{X^n - 1}$ and $\phi(x) = \eta x$, $\eta$ a primitive $n^{\mathrm{th}}$-root of 1.

Suppose $f(x)$ has nonzero residues at $\alpha_1, \ldots, \alpha_n$.
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Thus there are at most $n(n-1)$ $\phi$ permuting $\alpha_1, \ldots, \alpha_n$.

**Corollary**

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This can be sharpened from $n(n-1)$ to $n$ with more detailed analysis.

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3) Generic fibers $z' = g(z) + c$ are orthogonal to the constants; example if $f(z) = z^2 + az + b$ where $a$ and $b$ are algebraically independent over $\mathbb{Q}$, then generic fibers are orthogonal and orthogonal to $C$.

Thank you