Boolean algebras and semi-retractions

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Outline

1. Background
2. Semi-retractions
3. Categorical perspective

Ages

- Given an L-structure $\mathcal{M}$, age($\mathcal{M}$) is the class of all finitely generated substructures of $\mathcal{M}$ closed under isomorphism.
- $\mathcal{M}$ is **ultrahomogeneous** if any isomorphism between finitely generated substructures of $\mathcal{M}$ extends to an automorphism of $\mathcal{M}$.
- Fraïssé's theorem: For any countable signature L and countable (up to $\cong$) nonempty class $\mathcal{K}$ of finitely generated L-structures satisfying HP, JEP and AP, there is a unique (up to $\cong$) countable structure $\mathcal{F} = \mathrm{Flim}(\mathcal{K})$ such that $\mathcal{F}$ is ultrahomogeneous and $\mathrm{age}(\mathcal{F}) = \mathcal{K}$.
- For any countable structure $\mathcal{A}$ such that $\mathrm{age}(\mathcal{A}) \subseteq \mathcal{K}$, there is an embedding of $\mathcal{A}$ in $\mathcal{F}$.

Ramsey property (RP)

- For structures A and C, let $\binom{C}{A}$ denote all substructures $A' \subseteq C$ such that $A' \cong A$.
- A k-coloring c of $\binom{C}{A}$ is any function $c : \binom{C}{A} \to k$.
- By $C \to (B)^A_k$ we mean that for any k-coloring c of $\binom{C}{A}$, there is $B' \in \binom{C}{B}$ such that for any $A', A'' \in \binom{B'}{A}$, $c(A') = c(A'')$.

Definition

We say that a class $\mathcal{K}$ of finitely-generated L-structures has the **Ramsey property (RP)** if for all $A, B \in \mathcal{K}$ and integers $k \geq 2$ there exists $C \in \mathcal{K}$ such that $C \to (B)^A_k$

We say that B' is a copy of B **homogeneous for c (on copies of A)**.

- Given a structure $\mathcal{A}$, we say that $\mathcal{A}$ **has RP** if age($\mathcal{A}$) has RP.
- Working in $\mathcal{A}$ allows us to sweep some compactness arguments under the rug, e.g. show that for all A, B, k: $\mathcal{A} \to (B)^A_k$ (to show RP: $\mathcal{A}$).

Examples

- RP: All finite sets in $L = \emptyset$.
  RP: All finite linear orders in $L = \{<\}$. (order forgetful)
- $\neg$ RP: All finite simple graphs with no loops in $L = \{E\}$
  RP: All finite simple graphs with no loops with any ordering on the vertices in $L = \{E, <\}$. (not order forgetful)
- RP: Convexly ordered finite equivalence relations in $L = \{E, <\}$.
  $\neg$ RP: Finite equivalence relations with any ordering on points in $L = \{E, <\}$.
- RP: Finite Boolean algebras in $L = \{\vee, \wedge, \neg, \mathbf{0}, \mathbf{1}\}$ (Graham-Rothschild, '71)
  RP: Finite Boolean algebras with natural orders in $L = \{\vee, \wedge, \neg, \mathbf{0}, \mathbf{1}, <\}$ (Kechris-Pestov-Todorcevic, '05) (order forgetful)
- RP: $\mathrm{age}(\langle \mathbb{Z}, p, s \rangle)$
  $\neg$ RP: $\mathrm{age}(\langle \mathbb{Z}, s \rangle)$
- board #3 ...

Rigidity

- Rigidity can be accomplished with a definable linear order:

Definition

We say that a structure A is **rigid** if the only automorphism of A is the identity map.

Proposition

If age($\mathcal{B}$) consists of rigid elements, then for any $C, C' \in$ age($\mathcal{B}$), if $C \cong_{\mathcal{B}} C'$, then this is witnessed by a unique isomorphism $\tau : C \to C'$.

Semi-retractions

- Given length-n sequences $\bar{\imath}, \bar{\jmath}$ from some structure $\mathcal{M}$, by

$$\bar{\imath} \sim_{\mathcal{M}} \bar{\jmath}$$

  we mean that $\text{qftp}^{\mathcal{M}}(\bar{\imath}) = \text{qftp}^{\mathcal{M}}(\bar{\jmath})$.

- Given any structures $\mathcal{A}, \mathcal{B}$, we say that an injection $h : \mathcal{A} \to \mathcal{B}$ is **qftp-respecting** if for all finite, same-length tuples $\bar{\imath}, \bar{\jmath}$ from $\mathcal{A}$,

$$\bar{\imath} \sim_{\mathcal{A}} \bar{\jmath} \Rightarrow h(\bar{\imath}) \sim_{\mathcal{B}} h(\bar{\jmath}).$$

- Not to be mysterious, $h(\bar{\imath}) := (h(i_0), \ldots, h(i_{n-1}))$.

Definition

Let $\mathcal{A}$, $\mathcal{B}$ be any structures. We say that $\mathcal{A}$ is a **semi-retract of $\mathcal{B}$ (via (g, f))** if

1. there exist qftp-respecting injections: $\mathcal{A} \xrightarrow{g} \mathcal{B} \xrightarrow{f} \mathcal{A}$
2. such that: $\mathcal{A} \xrightarrow{fg} \mathcal{A}$ is an embedding

Say (g, f) is a **semi-retraction between $\mathcal{A}$ and $\mathcal{B}$.**

- board #1...

Terminology history

- From [Ahlbrandt and Ziegler(1986)]:

Definition

Given countable, $\aleph_0$-categorical structures $\mathcal{A}$ and $\mathcal{B}$, $\mathcal{A}$ is a **retraction** of $\mathcal{B}$ if there exist interpretations $f : \mathcal{A} \rightsquigarrow \mathcal{B}$ $g : \mathcal{B} \rightsquigarrow \mathcal{A}$ such that $g \circ f$ is homotopic to the identity interpretation on $\mathcal{A}$.

Theorem (T. Coquand)

Given countable $\aleph_0$-categorical structures $\mathcal{A}$ and $\mathcal{B}$, $\mathcal{A}$ is a retraction of $\mathcal{B}$ iff there are continuous homomorphisms

$$Aut(\mathcal{A}) \xrightarrow{\varphi} Aut(\mathcal{B}) \xrightarrow{\psi} Aut(\mathcal{A})$$

such that $\psi \circ \varphi = 1$.

- In contrast, semi-retraction maps are pointwise on the underlying sets:

$$\mathcal{A} \xrightarrow{g} \mathcal{B} \xrightarrow{f} \mathcal{A}$$

  such that fg is an embedding.
- board #2...

Reducts

- Reducts of structures with RP do not necessarily have RP (could lose AP, rigidity, ...)
- We will use this definition:

Definition

We say that $\mathcal{A}$ is a **quantifier-free reduct** of $\mathcal{B}$ if $|\mathcal{A}| = |\mathcal{B}| = M$ and $\sim_{\mathcal{B}}$ refines $\sim_{\mathcal{A}}$ on M, i.e. for all finite same-length tuples $\bar{\imath}, \bar{\jmath}$ from $|\mathcal{A}|$, $\bar{\imath} \sim_{\mathcal{B}} \bar{\jmath} \Rightarrow \bar{\imath} \sim_{\mathcal{A}} \bar{\jmath}$.

- Note: $\mathcal{A}$ is a quantifier-free reduct of $\mathcal{B}$ if and only if $|\mathcal{A}| = |\mathcal{B}|$ and the identity map $id : \mathcal{B} \to \mathcal{A}$ is qftp-respecting.
- We say that $\mathcal{A}, \mathcal{B}$ are **quantifier-free interdefinable** if $|\mathcal{A}| = |\mathcal{B}|$ and each of $\mathcal{A}$, $\mathcal{B}$ is a quantifier-free reduct of the other.
- Note: If $\mathcal{A}, \mathcal{B}$ are quantifier-free interdefinable, then the identity maps between $\mathcal{A}, \mathcal{B}$ give a semi-retraction (in either order).

Results

- Previous results

Theorem ([Scow(2021)])

Let $\mathcal{A}$ and $\mathcal{B}$ be any structures. Suppose that $\mathcal{A}$ is a semi-retract of $\mathcal{B}$. Furthermore, suppose that $\mathcal{B}$-indexed indiscernible sets have the modeling property. Then $\mathcal{A}$-indexed indiscernible sets have the modeling property.

Corollary

Let both $\mathcal{A}$ and $\mathcal{B}$ be locally finite ordered structures. Suppose that $\mathcal{A}$ is a semi-retract of $\mathcal{B}$ and $\mathcal{B}$ has RP. Then $\mathcal{A}$ has RP.

- Newer results

Theorem ([Bartošová and Scow()])

Let $\mathcal{A}, \mathcal{B}$ be structures in any signatures and suppose that $\mathcal{A}$ is a semi-retract of $\mathcal{B}$. Suppose that $\mathcal{A}$ is locally finite and age($\mathcal{B}$) consists of rigid elements. If $\mathcal{B}$ has RP, then $\mathcal{A}$ has RP.

- The rigidity and local finiteness assumptions can be dropped if both $\mathcal{A}$ and $\mathcal{B}$ are in relational signatures.

Nonlocally finite example

- $\mathcal{A} = (\mathbb{Z}, s)$ (not locally finite) is a semi-retract of $\mathcal{B} = (\mathbb{Z}, s, p)$ (non-rigid age)
- The identity maps on the underlying sets give a semi-retraction ($\mathcal{A}$, $\mathcal{B}$ are qf-interdefinable):

  $$\mathcal{A} \xrightarrow{id} \mathcal{B} \xrightarrow{id} \mathcal{A}$$

  The transfer theorem does not apply because both $\mathcal{A}$ and $\mathcal{B}$ fail to satisfy the conditions.
- **RP:** $\mathcal{B}$
  RP: $\mathcal{B}$**,** $\neg$ **RP:** $\mathcal{A}$**.**
- board #4...
- A slight modification gives an example on $\mathbb{N}$:
- $\mathcal{A}' = (\mathbb{N}, p)$ define $p(0) = 0$ (locally finite + rigid age) is a semi-retract of $\mathcal{B}' = (\mathbb{N}, s, p)$ (rigid age)
- The identity maps on the underlying sets give a semi-retraction ($\mathcal{A}'$, $\mathcal{B}'$ are qf-interdefinable):

  $$\mathcal{A}' \xrightarrow{id} \mathcal{B}' \xrightarrow{id} \mathcal{A}'$$

- **RP:** $\mathcal{B}' \Rightarrow$ **RP:** $\mathcal{A}'$**.**
  RP: $\mathcal{B}'$ (trivially)
  RP: $\mathcal{A}'$ **!!**
- Fun question: what is the Fraïssé limit of age($\mathcal{A}'$)? It should have one.

Boolean algebras

- If A is a finite Boolean algebra, a linear order $<_A$ on A is **natural** if it is the antilexicographic order on A induced by some linear order on the atoms of A.
- The ordered Boolean algebra $(\mathcal{B}_{ba}, \prec)$ is the Fraïssé limit of the class of finite Boolean algebras with natural linear orders. We refer to $\prec$ as a **normal** order on $\mathcal{B}_{ba}$.
- We say that $B = \{b_i : i < \omega\}$ is an **antichain** if the elements are pairwise disjoint, i.e. $b_i \wedge b_j = \mathbf{0}$ for every $i \neq j$.

Ordered and unordered

Theorem

$(\mathcal{R}, <)$ (locally finite) is a semi-retract of $(\mathcal{B}_{ba}, \prec)$ (rigid age).

- By the transfer theorem, RP should transfer from $(\mathcal{B}_{ba}, \prec)$ to $(\mathcal{R}, <)$.
- RP: $(\mathcal{B}_{ba}, \prec)$
  RP: $(\mathcal{R}, <)$

Theorem

$\mathcal{A} := \mathcal{R}$ (locally finite) is a semi-retract of $\mathcal{B} := \mathcal{B}_{ba}$ (non-rigid age).

- The transfer theorem does not apply because $\mathcal{B}$ fails to satisfy the rigidity condition.
- RP: $\mathcal{B}_{ba}$
  $\neg$ RP: $\mathcal{R}$

Categorical notions

- In [Mašulović and Scow(2017)] we determined that adjunctions transfer RP. Mašulović has found a more general notion that is sufficient to transfer RP:

Definition

Let $\mathbf{C}$ and $\mathbf{D}$ be categories and let $F : \mathrm{Ob}(\mathbf{D}) \rightleftarrows \mathrm{Ob}(\mathbf{C}) : G$ be maps on objects. We say that (F, G) is a **pre-adjunction** if for every $A \in \mathrm{Ob}(\mathbf{D})$ and $C \in \mathrm{Ob}(\mathbf{C})$ we have a map

$$\Phi_{A,C} : \hom_{\mathbf{C}}(F(A), C) \to \hom_{\mathbf{D}}(A, G(C)),$$

such that

$$\forall A, B \in \mathrm{Ob}(\mathbf{D})\ \forall C \in \mathrm{Ob}(\mathbf{C})\ \forall v \in \hom_{\mathbf{D}}(A, B)\ \forall \psi \in \hom_{\mathbf{C}}(F(B), C)$$

$$\exists w \in \hom_{\mathbf{C}}(F(A), F(B)) \text{ such that } \Phi_{A,C}(\psi \circ w) = \Phi_{B,C}(\psi) \circ v.$$

Theorem ([Mašulović(2018)])

*If $\mathbf{C}$ has the Ramsey property ("for morphisms") and $F : Ob(\mathbf{D}) \rightleftarrows Ob(\mathbf{C}) : G$ is a **pre-adjunction**, then $\mathbf{D}$ has the Ramsey property ("for morphisms").*

- If an age $\mathcal{K}$ consists of rigid elements and we consider $\mathrm{Ob}(\mathbf{C}) := \mathcal{K}$ and embeddings as morphisms then the above Ramsey property ("for morphisms") is equivalent to the RP as we defined it.

Pre-adjunctions

- semi-retractions ⇒ pre-adjunctions

Theorem

Any semi-retraction (g, f) between $\mathcal{A}$ and $\mathcal{B}$ defines a pre-adjunction between the categories of finite tupes of $\mathcal{A}$ and B, respectively, with qftp-preserving injections.

- More specifically, in the category in which we determine the RP:

Theorem

Let $\mathcal{A}$ and $\mathcal{B}$ be locally finite and let (g, f) be a semi-retraction between $\mathcal{A}$ and $\mathcal{B}$. Then there is a pre-adjunction between age($\mathcal{A}$) and age($\mathcal{B}$) with embeddings as morphisms.

- Question: can any pre-adjunction be understood somehow as pointwise maps, as in the case of semi-retractions?

When are transfers witnessed by semi-retractions

- Among reducts of a structure with RP: semi-retractions characterize those with RP, under simplifying assumptions for $\mathcal{B}$:

Theorem

Fix locally finite ordered structures $\mathcal{A}$ and $\mathcal{B}$ and suppose that $\mathcal{A}$ is a quantifier-free reduct of $\mathcal{B}$ and $\mathcal{B}$ is saturated.

Suppose that there are only finitely many quantifier-free n-types in $\mathcal{B}$ for any $n \geq 1$.

Suppose that $\mathcal{B}$ has RP. Then, $\mathcal{A}$ is a semi-retract of $\mathcal{B}$ if and only if $\mathcal{A}$ has RP.

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