# A first-order definition for Campana points in $\mathbb{Q}$

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# Introduction

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Given any field $K$ and a natural number $n$, we say that a given set $A \subseteq K^n$ is *diophantine over* $K$, or *first-order existentially defined over* $K$ (or simply "existentially defined") if there exists $m \in \mathbb{N}$ and $P \in K[X_1, \cdots, X_m, Y_1, \cdots, Y_n]$ such that, for any $a = (a_1, \cdots, a_n) \in K^n$, we have $a \in A$ if and only if there exist $x_1, \cdots, x_m \in K$ such that $P(x_1, \cdots, x_m, a_1, \cdots, a_n) = 0$.

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If $\mathbb{Z}$ were diophantine over $\mathbb{Q}$, $\mathrm{Th}_{\exists}(\mathbb{Z})$ would be interpretable in $\mathrm{Th}_{\exists}(\mathbb{Q})$, yielding undecidability of $\mathrm{Th}_{\exists}(\mathbb{Q})$.

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## Koenigsmann, 2010

There exist $n \in \mathbb{N}$ and $g \in \mathbb{Z}[t, x_1, \cdots, x_n]$ such that, given $t \in \mathbb{Q}$, we have $t \in \mathbb{Z}$ if and only if

$$\mathbb{Q} \models \forall x_1 \cdots \forall x_n \left(g\left(t, x_1, \cdots, x_n\right) \neq 0\right)$$

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The two last results are clever generalizations of the same techniques and methods developed by Koenigsmann in 2010.

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Complexity was also measured.

**Geng-Rui Zhang & Zhi-Wei Sun, 2021**

The universal definition for $\mathbb{Z}$ in $\mathbb{Q}$ can be taken to involve a polynomial with 32 unknowns and degree at most $6 \cdot 10^{11}$.

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- Jochen Koenigsmann. “Defining $\mathbb{Z}$ in $\mathbb{Q}$”. In: *Ann. of Math. (2)* 183.1 (2016), pp. 73– 93. ISSN: 0003-486X. doi : 10.4007/annals.2016.183.1.2.

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- Campana points.

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Given $n \in \mathbb{N}$, the set $R_n$ of $n$-Campana points is the set of all rational numbers $r \in \mathbb{Q}$ such that either $r = 0$ or $r \neq 0$ and for all $p \in \mathbb{P}$, $\nu_p\left(r^{-1}\right) \in \{0\} \cup \mathbb{N}_{\geq n}$.

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**Question**

Can we find a good first-order definition for Campana points?

# Preliminary facts

## Definition

Given a field $k$, the (quadratic) *Hilbert Symbol* is the function $(-,-) : k^\times \times k^\times \to \{\pm 1\}$ defined as

$$(a,b) = \begin{cases} 1, & z^2 - ax^2 - by^2 = 0 \text{ has a nontrivial solution,} \\ -1, & \text{otherwise.} \end{cases}$$

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There are two very important results in Number Theory about Hilbert symbols. First, there is an explicit formula, in terms of Legendre Symbols, to compute Hilbert symbols. Second, a local-to-global principle.

# Preliminary facts

## Computation of Hilbert symbols

If $p \in \mathbb{P} \cup \{\infty\}$ and $a, b \in \mathbb{Q}_p^\times$, define $u_\ell := \frac{a}{\ell^{\nu_\ell(a)}} \in \mathbb{Z}_\ell^\times$ and $v_\ell := \frac{b}{\ell^{\nu_\ell(b)}} \in \mathbb{Z}_\ell^\times$ for each $\ell \in \mathbb{P}$. Then we have:

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If $p = \infty$, then $(a, b)_\infty = -1$ if and only if $a < 0$ and $b < 0$. Otherwise, $(a, b)_\infty = 1$.

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If $p = 2$, then

$$(a, b)_2 = (-1)^{\frac{u_2 - 1}{2} \times \frac{v_2 - 1}{2} + \nu_2(a) \frac{v_2^2 - 1}{8} + \nu_2(b) \frac{u_2^2 - 1}{8}}$$

(here the exponents are replaced by an element of $\mathbb{Z}/2\mathbb{Z}$ via reduction modulo 2).

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If $p \in \mathbb{P} \setminus \{2\}$,

$$(a, b)_p = \underbrace{\left(\frac{u_p}{p}\right)^{\nu_p(b)} \left(\frac{v_p}{p}\right)^{\nu_p(a)}}_{\text{Legendre symbols}} (-1)^{\nu_p(a)\nu_p(b)\frac{p-1}{2}}.$$

# Preliminary facts

## Local-to-global principle

Let $I$ be a finite set and fix $\{a_i\}_{i \in I} \subseteq \mathbb{Q}^\times$. For each $(i, v) \in I \times (\mathbb{P} \cup \{\infty\})$ fix $\varepsilon_{i,v} \in \{-1, 1\}$. The following are equivalent:

- There exists $x \in \mathbb{Q}^\times$ such that $(a_i, x)_v = \varepsilon_{i,v}$ for all $(i, v) \in I \times (\mathbb{P} \cup \{\infty\})$.
- All these conditions hold:
  1. $\varepsilon_{i,v} = 1$ for all but finitely many $(i, v) \in I \times (\mathbb{P} \cup \{\infty\})$.
  2. For all $i \in I$ we have $\displaystyle\prod_{v \in \mathbb{P} \cup \{\infty\}} \varepsilon_{i,v} = 1$.
  3. For all $v \in \mathbb{P} \cup \{\infty\}$ there exists $x_v \in \mathbb{Q}^\times$ such that $(a_i, x_v)_v = \varepsilon_{i,v}$ for all $i \in I$.

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- $J_{a,b} = \left(I_{a,b}^{a} + I_{a,b}^{a}\right) \cap \left(I_{a,b}^{b} + I_{a,b}^{b}\right)$.

## Some facts

- Given $a, b \in \mathbb{Q}^{\times}$ and $p \in \mathbb{P} \cup \{\infty\}$, we have $p \in \Delta_{a,b}$ if and only if $(a, b)_p = -1$.

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- $J_{a,b} = \left(I_{a,b}^{a} + I_{a,b}^{a}\right) \cap \left(I_{a,b}^{b} + I_{a,b}^{b}\right)$.

## Some facts

- Given $a, b \in \mathbb{Q}^{\times}$ and $p \in \mathbb{P} \cup \{\infty\}$, we have $p \in \Delta_{a,b}$ if and only if $(a, b)_p = -1$.
- $\{(a, b, r) \in \mathbb{Q}^{\times} \times \mathbb{Q}^{\times} \times \mathbb{Q} : r \in J_{a,b}\}$ is diophantine over $\mathbb{Q}$.

# A parametrization of the finite sets of primes

For each $a, b \in \mathbb{Q}^{\times}$ we define

$$\Delta^{a,b} := \begin{cases} \Delta_{a,b} \setminus \{2, \infty\}, & 2 \in \Delta_{a,b} \text{ and } \nu_2(a), \nu_2(b) \text{ are even,} \\ \Delta_{a,b} \setminus \{\infty\}, & \text{else.} \end{cases}$$

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If $a, b \in \mathbb{Q}^{\times}$ then $J_{a,b} = \bigcap_{p \in \Delta^{a,b}} p\mathbb{Z}_{(p)}$.

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If $a, b, c, d \in \mathbb{Q}^{\times}$ then $J_{a,b} + J_{c,d} = \bigcap_{p \in \Delta^{a,b} \cap \Delta^{c,d}} p\mathbb{Z}_{(p)}$.

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## Corollary

$\{(a, b, c, d) \in \mathbb{Q}^\times \times \mathbb{Q}^\times \times \mathbb{Q}^\times \times \mathbb{Q}^\times : \Delta^{a,b} \cap \Delta^{c,d} = \emptyset\}$ is a diophantine subset of $\mathbb{Q} \times \mathbb{Q} \times \mathbb{Q} \times \mathbb{Q}$.

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*Idea of the proof:* For $S = \emptyset$ take $a = b = 1$. Otherwise, define $a := (-1)^{|S|} \prod_{p \in S} y_p$. We want to find $b \in \mathbb{Q}^\times$ such that $(a, b)_p = -1$ if and only if $p \in S$. By the local-to-global principle, it suffices to find, for each $v \in \mathbb{P} \cup \{\infty\}$, some $b_v$ with $(a, b_v) = -1$ if $v \in S$ and $(a, b_v) = 1$ if $v \in \mathbb{P} \setminus S$.

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By the formula for $(-,-)_p$, this reduces to a clever Chinese Remainder Theorem construction (use the archimedean place to fix signs and get the technical conditions in the local-to-global principle).

# Some additional observations

Remember: $\{(a, b, r) \in \mathbb{Q}^\times \times \mathbb{Q}^\times \times \mathbb{Q} : r \in J_{a,b}\}$ is diophantine over $\mathbb{Q}$.

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For each $n \in \mathbb{N}$ define

$$J_{a,b,n} := \prod_{i=1}^{n} J_{a,b} \text{ (set-theoretically)}$$

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## Proposition

$\{(a,b,r) \in \mathbb{Q}^\times \times \mathbb{Q}^\times \times \mathbb{Q} : r \in J_{a,b,n} \setminus \{0\}\}$ and
$\left\{(a,b,r) \in \mathbb{Q}^\times \times \mathbb{Q}^\times \times \mathbb{Q} : r \in (J_{a,b,n} \setminus \{0\})^{-1}\right\}$ are diophantine over $\mathbb{Q}$.

# Completing our first aim

**Theorem**

The set $\left\{(a, b, r) \in \mathbb{Q}^{\times} \times \mathbb{Q}^{\times} \times \mathbb{Q} : r \text{ is integral outside } \Delta^{a,b}\right\} \cup (\{0\} \times \mathbb{Q} \times \mathbb{Q}) \cup (\mathbb{Q} \times \{0\} \times \mathbb{Q})$ is universal in $\mathbb{Q} \times \mathbb{Q} \times \mathbb{Q}$.

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*Proof.* A defining formula for this set is

$$\forall c \forall d \left( \left[ abcd \neq 0 \wedge \Delta^{a,b} \cap \Delta^{c,d} = \emptyset \right] \Rightarrow r \notin \left( J_{c,d} \setminus \{0\} \right)^{-1} \right)$$

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This proves what we want. ∎

If you want to get the original result for a specific $S$, just use the above formula, but instead of letting $a$ and $b$ be unknowns, let them be some fixed elements of $\mathbb{Q}^\times$ such that $\Delta^{a,b} = S$.

# Completing our first aim

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I repeated the above proof trying to be as explicit as I can to find the number of unknowns and a bound for the degree of the polynomial in the formula. By doing so, I attained this more specific statement:

**Theorem**

There exists $P \in \mathbb{Z}[A, B, R, X_1, \cdots, X_{250}]$ of degree at most 128 such that, for any $a, b, r \in \mathbb{Q}$, the following are equivalent:

- $ab = 0$ or $ab \neq 0$ and $r$ is integral outside $\Delta^{a,b}$.
- For all $x_1, \cdots, x_{250} \in \mathbb{Q}$ we have $P(a, b, r, x_1, \cdots, x_{250}) \neq 0$.

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We can turn this into a $\forall\exists$-definition by taking our boxed universal and turning it into a $\forall\exists$; because $x \neq 0$ is the same as $\exists y\,(xy = 1)$.

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This is not as easy as it seems. The generalization is straightforward for number fields with at least one real infinite place (remember that we needed such a place to make a sign correction). For number fields with no real infinite places, it seems that a more clever use of the Chinese Remainder Theorem will be needed.

# The end

THANK YOU