Forcing in Algebraic Field Extensions of the Rationals

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Intuition for "general tendency"

The equation $X^5 + Y^5 = 1$ has no nonzero solutions in $\mathbb{Q}$. However, it has plenty of solutions in $\overline{\mathbb{Q}}$, and if we choose a subfield F of $\overline{\mathbb{Q}}$ "at random," it seems near-certain that F will contain such a solution.

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More rigorously: no matter which (finitely many) elements have already been included in F or excluded from F, there will still remain infinitely many solutions in $\overline{\mathbb{Q}}$ that could yet appear in F.

(Indeed, for infinitely many $q \in \mathbb{Q}$, $\sqrt[5]{1 - q^5}$ could yet appear, and each of these has degree 5 over $\mathbb{Q}$.)

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... but this situation is different! Suppose that, in dividing up the elements of $\overline{\mathbb{Q}}$, we decide that $\sqrt{2} \notin F$. Then F cannot contain any nonzero solution, because if $x^2 - 2y^2 = 0 \neq xy$ and $x, y \in F$, then $\frac{x}{y} \in F$, yet $(\frac{x}{y})^2 = 2$.

Thus the choice of excluding $\sqrt{2}$ from F ruled out all nonzero solutions (whereas including $\sqrt{2}$ in F would immediately yield a solution). In this example, both the existential sentence and its negation

$$(\exists x, y)\ x^2 - 2y^2 = 0 \neq xy \qquad\qquad (\forall x, y)\ \neg(x^2 - 2y^2 = 0 \neq xy)$$

seem reasonably (equally?) likely to hold.

Topology on the subfields of $\overline{\mathbb{Q}}$

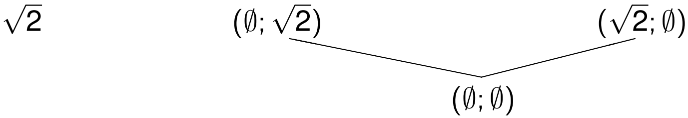
Fix one computable presentation $\overline{\mathbb{Q}}$ of the algebraic closure of $\mathbb{Q}$. Each choice of finitely many elements constitutes a *condition* on subfields. We write $(\overline{a}; \overline{b})$ to denote the condition saying that all of $\overline{a}$ is included and all of $\overline{b}$ is excluded. Then the set

$$\mathcal{U}_{\overline{a};\overline{b}} = \{F \subseteq \overline{\mathbb{Q}} : \mathbb{Q}(\overline{a}) \subseteq F \ \&\ F \cap \{\overline{b}\} = \emptyset\}$$

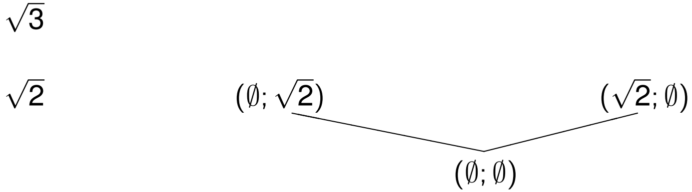
is a basic open set in our topology on the space **Sub**($\overline{\mathbb{Q}}$) of all subfields of $\overline{\mathbb{Q}}$, and the topology is generated by these basic open sets, as $\overline{a}$ and $\overline{b}$ range over all finite tuples from $\overline{\mathbb{Q}}$.

The relations $\mathcal{U}_{\overline{a};\overline{b}} \subseteq \mathcal{U}_{\overline{c};\overline{d}}$, $\mathcal{U}_{\overline{a};\overline{b}} = \textbf{Sub}(\overline{\mathbb{Q}})$, and $\mathcal{U}_{\overline{a};\overline{b}} = \emptyset$ are decidable, by theorems of Kronecker.

Picture: the space Sub($\overline{\mathbb{Q}}$) of all subfields of $\overline{\mathbb{Q}}$



Picture: the space $\mathrm{Sub}(\overline{\mathbb{Q}})$ of all subfields of $\overline{\mathbb{Q}}$



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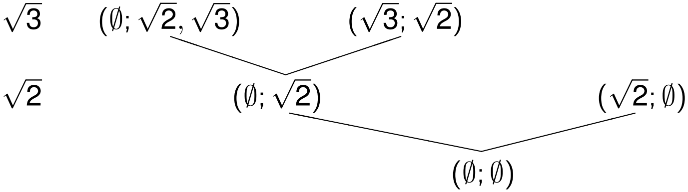

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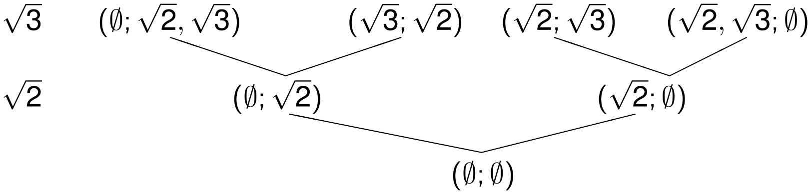

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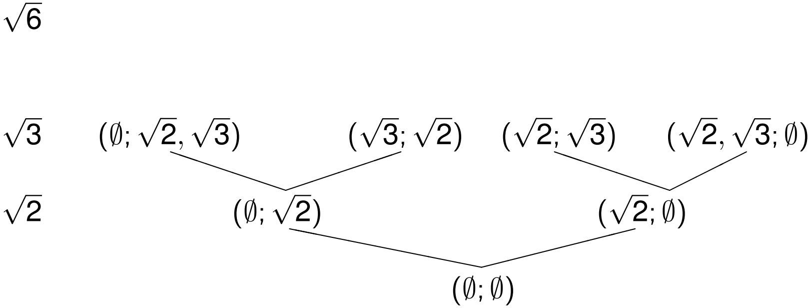

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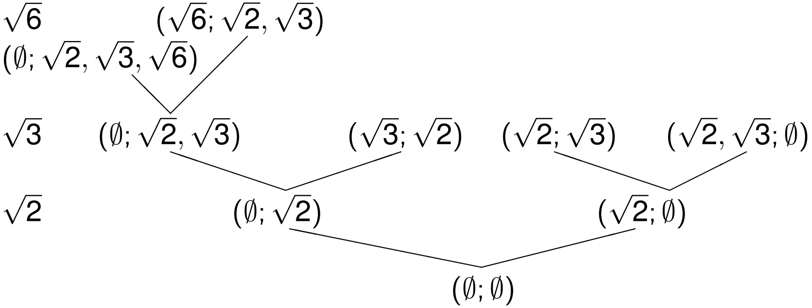

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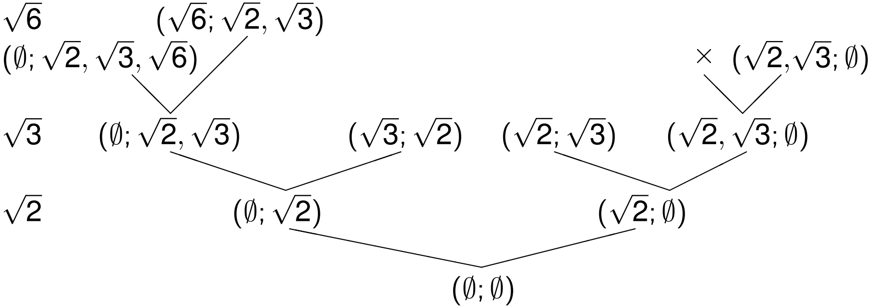

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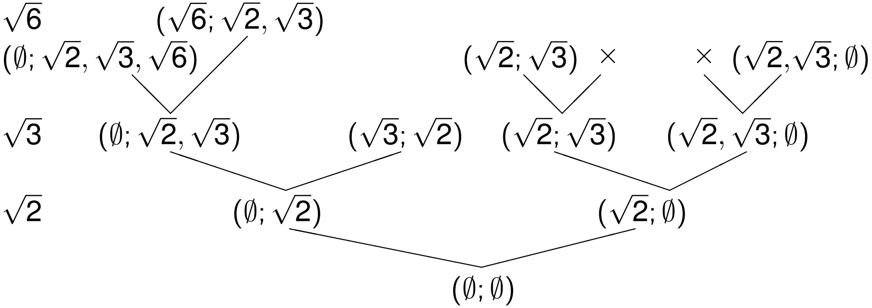

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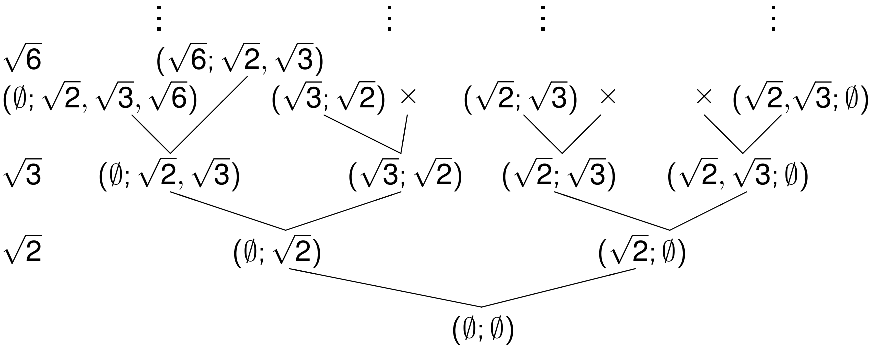

The nodes $\times$ are unsatisfiable conditions: if we have ruled out $\sqrt{2}$, then F cannot contain both $\sqrt{3}$ and $\sqrt{6}$. But we still get a decidable subtree of $2^{<\omega}$, with no terminal nodes and no isolated paths. So the set of paths through it is homeomorphic to Cantor space 2^{ω}. This is the space **Sub**($\overline{\mathbb{Q}}$), with each path naming a subfield.

Conditions and forcing

Definition

A condition $(\overline{a};\overline{b})$ *forces* a sentence φ, written $(\overline{a};\overline{b}) \Vdash \varphi$, if

$$\{F \in \mathcal{U}_{\overline{a};\overline{b}} : \varphi \text{ is true in } F\}$$

is dense within $\mathcal{U}_{\overline{a};\overline{b}}$ in our topology.

In our examples earlier:

- $(\emptyset; \sqrt{2}) \Vdash (\forall x \forall y) \neg [x^2 - 2y^2 = 0 \neq xy]$.
- $(\sqrt{2}; \emptyset) \Vdash (\exists x \exists y)\ x^2 - 2y^2 = 0 \neq xy$.
- $(\emptyset; \emptyset) \Vdash (\exists x \exists y)\ x^5 + y^5 - 1 = 0 \neq xy$.

Notice that in the third item, not all fields in $\mathcal{U}_{\emptyset;\emptyset}$ satisfy the sentence given – e.g., $\mathbb{Q}$ does not – but densely many of them satisfy it. Forcing does not quite guarantee the truth of the sentence being forced!

Specifics of forcing $\exists$ and $\forall$ sentences

If $(\overline{a}; \overline{b}) \Vdash \forall\vec{x}\neg\psi(\vec{x})$, then in fact every field in $\mathcal{U}_{\overline{a};\overline{b}}$ satisfies $\forall\vec{x}\neg\psi(\vec{x})$.
If any $F \in \mathcal{U}_{\overline{a};\overline{b}}$ contained a tuple $\overline{c}$ with $\psi(\overline{c})$, then $(\overline{a}, \overline{c};\ \overline{b})$ would be consistent (since F exists!) and every field in $\mathcal{U}_{\overline{a},\overline{c};\ \overline{b}}$ would contain this witness $\overline{c}$. Since $\mathcal{U}_{\overline{a},\overline{c};\ \overline{b}} \subseteq \mathcal{U}_{\overline{a};\overline{b}}$, this would contradict the density in $\mathcal{U}_{\overline{a};\overline{b}}$ of the fields satisfying $\forall\vec{x}\neg\psi(\vec{x})$.

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However, as seen with $X^5 + Y^5 = 1$ above, a condition can force an existential sentence without the sentence being true in all fields realizing the condition.

Key theorem

Theorem (Eisenträger, M, Springer, and Westrick)

It is decidable whether a condition $(\overline{a}; \overline{b})$ forces an existential or universal sentence φ (with parameters from $\mathbb{Q}(\overline{a})$). The decision procedure is uniform in $\overline{a}$, $\overline{b}$, and φ.

The proof is not simple. We show that whenever $(\overline{a}; \overline{b}) \Vdash \forall \vec{X} \neg \psi$, there is a "reason." For $X^2 - 2Y^2 = 0 \neq XY$, the reason was the rational function $\frac{X}{Y}$, which is a square root of 2 whenever $X^2 - 2Y^2 = 0 \neq XY$. Similarly, whenever $(\overline{a}; \overline{b}) \Vdash \forall \vec{X} \neg \psi$, there is some rational function $\frac{p(\vec{X})}{q(\vec{X})}$ such that whenever $\psi(\vec{x})$ holds, $\mathbb{Q}\left(\overline{a}, \frac{p(\vec{x})}{q(\vec{x})}\right)$ contains an element of $\overline{b}$.

However, to follow through on the reason, we need an argument by transfinite induction along an ordinal ranking of existential senences.

Why this helps

We now focus on the class of *generic* (specifically, 1-generic) fields. These fields form a comeager class in $\textbf{Sub}(\overline{\mathbb{Q}})$. So, in the sense of Baire category, a property that holds of all generic fields may be considered to hold "almost everywhere."

A field F is 1-generic if F lies in every $\emptyset'$-decidable dense open subset of $\textbf{Sub}(\overline{\mathbb{Q}})$. We actually need only a consequence of genericity: that if S is a computably enumerable set of conditions, and every condition $(\overline{a}, \overline{b})$ realized by F can be extended to a nontrivial condition $(\overline{a}, \overline{c};\ \overline{b}, \overline{d})$ in S, then F itself must realize some condition in S.

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To make sense of this, consider the following set S:

$$\{\text{conditions } (\overline{a}; \overline{b}) : \mathbb{Q}(\overline{a}) \text{ contains a solution to } X^5 + Y^5 - 1 = 0 \neq XY\}.$$

This S is c.e., and we know it is dense above the empty condition $(\emptyset; \emptyset)$. Therefore, every generic field F must realize a condition in S, and thus must satisfy $(\exists X, Y)\ X^5 + Y^5 - 1 = 0 \neq XY$.

Original result

Theorem (EMSW)

Let Z be a subset of $\mathbb{Q}$ that is neither cofinite nor thin there. Let $F \subseteq \overline{\mathbb{Q}}$ be a 1-generic field. Then Z is not universally definable in F. Indeed, no set S that is universally definable in F can have $S \cap \mathbb{Q} = Z$.

Dually, if $Y \subseteq \mathbb{Q}$ is neither finite nor co-thin in $\mathbb{Q}$, then no set S that is existentially definable in F can have $S \cap \mathbb{Q} = Y$.

Originally we wanted to show this for $Z = \mathbb{Z}$, which is neither thin nor co-thin in $\mathbb{Q}$. We concluded that in a generic field F, neither $\mathbb{Z}$ nor $\mathcal{O}_F$ can be existentially or universally definable.

Dittmann and Fehm subsequently extended this, showing that no proper subring of F can be defined in F by any first-order formula. The result above has no obvious extension to more complex definitions, but does hold even when Z is not a ring.

Newer results

Proposition

Let φ be an existential or universal sentence, and let $F \in \textbf{Sub}(\overline{\mathbb{Q}})$ be a generic field. Then φ holds in F iff F realizes some condition that forces φ. (In brief: $F \models \varphi$ iff F realizes an $(\overline{a}; \overline{b})$ with $(\overline{a}; \overline{b}) \Vdash \varphi$.)

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So, for a generic field F, we can decide the entire existential theory of F just by knowing which conditions F realizes.

In turn, the conditions realized by F can be determined if we know F as a subfield of $\overline{\mathbb{Q}}$, or equivalently, if we know the atomic diagram of F and the single-variable version of $HTP(F)$:

$$HTP_1(F) = R_F = \{g \in F[X] : (\exists x \in F)\; g(x) = 0\}.$$

What about $HTP_1(F)$?

Previous results determined the situation for generic F:

Theorem (M, 2020)

Let F be a generic algebraic extension of $\mathbb{Q}$. Then $HTP_1(F)$ must be *low* relative to the atomic diagram $D(F)$:

$$(HTP_1(F) \oplus D(F))' \equiv_T (D(F))'.$$

Also, there must exist copies $K, L \cong F$ such that $HTP_1(K) \not\leq_T D(K)$ while $HTP_1(L) \leq_T D(L)$.

One can argue that $HTP_1(K) \not\leq_T D(K)$ is the "common" situation for copies of F, with $HTP_1(L) \leq_T D(L)$ being exceptional.

General tendency of $HTP(F)$ for $F \subseteq \overline{\mathbb{Q}}$

Theorem (EMSW)

For all generic algebraic extensions F of $\mathbb{Q}$, the following sets are Turing-equivalent relative to $D(F)$:

- The root set $R_F = HTP_1(F)$.
- $HTP(F)$.
- The image of F in $\overline{\mathbb{Q}}$ under a ($D(F)$-computable) field embedding.

Moreover, all of these are of low Turing degree relative to $D(F)$, and in general they are not computable relative to $D(F)$ (although exceptional copies of F do exist).
Notice that therefore many sets that are $D(F)$-computably enumerable (including the Halting Problem itself) fail to be diophantine in F.

Since the generic extensions of $\mathbb{Q}$ form a comeager class in the space of all algebraic extensions, each of these properties may be considered to hold of “almost all” algebraic extensions of $\mathbb{Q}$, in the sense of Baire category.

$HTP^{\infty}(F)$

Let $HTP^{\infty}(F) = \{f \in F[\vec{X}] : f = 0$ has infinitely many solutions in $F\}$.

Theorem (EMSW)

It is decidable, uniformly in $\overline{a}$, $\overline{b}$, and f, whether $(\overline{a}, \overline{b}) \Vdash f \in HTP^{\infty}(F)$.

Corollary

For all 2-generic extensions F of $\mathbb{Q}$, $HTP^{\infty}(F) \equiv_T HTP_1(F)$ is again low (but in general noncomputable) relative to $D(F)$.

Corollary (of the proof)

For every condition $(\overline{a}; \overline{b})$, there exists a computable field $F \in \mathcal{U}_{\overline{a};\overline{b}}$ such that $HTP(F)$ and $HTP^{\infty}(F)$ are decidable.