On the Borel complexity of modules

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Study: The complexity of the class of countable models of a theory T in a countable language.

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Would like finer invariants.

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For T any L-theory, $Mod(T) = \{M \in Str_L : M \models T\}$ is a Borel subset.

How complicated is $(Mod(T), \cong)$?

Friedman-Stanley Given two theories T, S (possibly in different countable languages L, L') we say $(Mod(T), \cong)$ is Borel reducible to $(Mod(S), \cong)$, $T \leq_B S$, if there is a Borel $f : Str_L \to Str_{L'}$ such that for all $M, N \in Mod(T)$, $f(M), f(N) \in Mod(S)$ and

$$M \cong N \iff f(M) \cong f(N)$$

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Clearly, $T \leq_B S$ implies $I(T, \aleph_0) \leq I(S, \aleph_0)$, but Borel complexity can separate some theories with $I(T, \aleph_0) = 2^{\aleph_0}$.

Some examples

$Th(\mathbb{Z}, +)$, $REF(bin)$, RCF have $I(T, \aleph_0) = 2^{\aleph_0}$.

- $Th(\mathbb{Z}, +)$, $REF(bin)$ are $\leq_B$-incomparable, but both $<_B$ RCF.
- Isomorphism is Borel on $Mod(Th(\mathbb{Z}, +))$, but not on $Mod(REF(bin))$.

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There is a maximal $\equiv_B$-class, containing graphs, linear orders, RCF, DCF_0. A theory T is Borel complete if $(Mod(T),\cong)$ is in this maximal class.

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Borel completeness does not play nicely with the usual dividing lines: Some ω-stable, Morley rank 2 theories are Borel complete, as are some weakly minimal, trivial theories; DLO (or any ω-categorical theory) is $\leq_B$-minimal; $Th(\mathbb{Z},+)$ has a Borel complete reduct.

Modules

Fix a countable ring R (with unit). Let $L_R = \{+, 0, f_r : r \in R\}$. Study the class of countable left R-modules (typically an incomplete L_R theory).
(Ex: For $n \neq m$, $(\mathbb{Z}^n, +, 0)$ is not elementarily equivalent to $(\mathbb{Z}^m, +, 0)$.)

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Will see: For many (countable) rings R, Th(left R-modules) is Borel complete, but not always:

- If R is a finite field, then any two countably infinite R-modules (vector spaces) are isomorphic.
- If R is a countably infinite field, there are countably many non-isomorphic countable vector spaces.
- If R is a finite product of fields, then again, countably many countable R-modules.

The main theorem

Theorem (L-Ulrich)

Let R be any countable, commutative ring (with 1). Then

- *either $Th(R\text{-modules})$ is Borel complete; or*
- *There are only countably many isomorphism types of countable R-modules.*

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Conclusion: Borel complexity says nothing interesting for (incomplete) theories of commutative R-modules.

Corollary

(to proof) The class of torsion free abelian groups is Borel complete.

An easy Borel reduction

Fact: For any countable ring R (with 1), and for any 2-sided ideal $I \subseteq R$,

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Thus, if Th(left R/I-modules) is Borel complete, then so is Th(left R-modules).

Tagged R-modules

Fix a countable ring R (with 1), but not necessarily commutative. A tagged left R-module $\overline{V} = (V, V_n)_{n \in \omega}$ is a left R-module V with countably many named left R-submodules V_n. (R. Göbel)

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For any countable ring R, the class of left R-modules with 4 distinguished submodules is Borel complete.

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For any countable ring R, (even if R is a finite field) the class of left R-modules with 4 distinguished submodules is Borel complete.

Model theoretic construction

Let **K** be a class of countable structures A such that:

- **K** is closed under $\cong$;
- Every $A \in$ **K** is finitely generated (there is some finite $X \subseteq A$ so that $cl(X) = A$);
- Some $A \in$ **K** is $\emptyset$-generated and every $A \in$ **K** has a proper extension;
- **K** has disjoint amalgamation.

Fact: For any such **K**, there is a **K**-limit $M = \bigcup_i A_i$ generated by X and an equivalence relation $E \subseteq X^2$ such that X/E is infinite and absolutely indiscernible

- Every $h \in Sym(X/E)$ lifts to an automorphism σ of M, i.e., for all $a \in X$, $h(a/E) = \sigma(a)/E$.

Example: Let **K** be the class of all finite, tagged $\mathbb{F}_2$-vector spaces $(A, V_n)_{n\in\omega}$ satisfying:

- $X := A \setminus \bigcup\{V_n : n \in \omega\}$ is a basis for A;
- $V_n = \{0\}$ for all but finitely many $n \in \omega$.

Here, **K** has only countably many isomorphism types, and M can be taken as the Fraïssé limit of **K**.

- The universe M and each V_n^M are isomorphic to $\bigoplus_\omega \mathbb{F}_2$.
- $X = M \setminus \bigcup\{V_n^M : n \in \omega\}$, X is a basis for M, but X is not indiscernible.
- There is an equivalence relation $E \subseteq X^2$ such that X/E is infinite and absolutely indiscernible.

Proof of Theorem 1

Using this: Given any countable R, construct a single countable tagged left R-module $\overline{N} = (N, V_n)_{n \in \omega}$ with a highly controlled automorphism group, and distinguished $Aut(\overline{N})$-invariant X, E with X/E absolutely indiscernible.

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To prove that Th(left R-modules) is Borel complete:

Given a countable graph $G = (X/E, R^G)$, construct a left R-submodule $U_G \leq N$ so that

$$G \mapsto N_G := (\overline{N}, U_G)$$

is a Borel reduction.

From tagged R-modules to R-modules

Warm-up: A tagged R-module $\overline{V} = (V, V_n)_{n\in\omega}$ is free-like if $V = \bigoplus_\omega R$, every $V_n \cong \bigoplus_\omega R$, and $V/V_n \cong \bigoplus_\omega R$.

Lemma

(tagged left R-modules) $\leq_B$ (free-like tagged left R modules).

Thus, for any R, (free-like tagged left R modules) is Borel complete.

Theorem

If a countable R has a 'defect' then there is a Borel reduction

$$\text{free-like tagged left } R\text{-modules} \leq_B \text{ left } R\text{-modules}$$

Three 'defects'

A. If there is a central, non-zero divisor, non-unit $r \in R$, then Th(left R-modules) is Borel complete.

B. If there are central $r, s \in R$ with $(r) \cap (s) = 0$ and $I = Ann(r) + Ann(s)$ proper $(1 \notin I)$ then Th(left R-modules) is Borel complete.

C. If R is a commutative ring with a strictly descending chain of annihilator ideals $\{Ann(X_n) : n \in \omega\}$ with $\bigcap_{n \in \omega} Ann(X_n) = 0$, then Th($R$-modules) is Borel complete.

Fact: If R is a countable, commutative ring such that no R/I satisfies A,B or C, then R is an artinian principal ideal ring.

1. If any R/I is an integral domain that is not a field, then A.
2. Thus, we may assume every prime ideal is maximal.
3. If there are infinitely many prime (maximal) ideals, then by a Ramsey argument, get C. holding in some quotient.
4. Thus, there are only finitely many prime ideals, and the Jacobson radical=Nil-radical.
5. It follows that R is a finite product of local rings of the form Re for some idempotent e.
6. Fix one of the factors with maximal ideal $\mathbf{m}$. If the ideals $\subseteq \mathbf{m}$ are not linearly ordered, then get B. in some quotient.
7. If a factor is not Noetherian, then get a descending sequence of annihilator ideals, giving C.
8. Thus, in each factor there are only finitely many ideals, each principal.

Sketch of A.

Fix $r \in R$ central, non-zero divisor, non-unit. Think about r-adics

$\langle (r^m) : m \geq 1 \rangle$ is strictly decreasing. If $I = \bigcap_m (r^m) \neq 0$, then work in R/I. So we may assume $\bigcap_m (r^m) = 0$.

Let $\hat{R} = \varprojlim R/(r^m)$. For each $s \subseteq \omega$ finite, let $\sigma_s = \sum_{i \in s} r^i$ and let

$$\Gamma = \varprojlim \{\sigma_s : s \subseteq \omega \text{ finite}\}$$

Find $\{\gamma_n : n \in \omega\} \subseteq \Gamma$ algebraically independent, i.e., $p(\gamma_0, \ldots, \gamma_n) \neq 0$ for all non-constant $p \in R[x_0, \ldots, x_n]$. (Baire category)

Sketch of A. (continued)

Code: Given a free-like tagged left R-module $\overline{V} = (\bigoplus_\omega R, V_n)_{n\in\omega}$, let $f(\overline{V})$ be the smallest r-pure R-submodule of $\bigoplus_\omega \hat{R}$ generated by

$$\bigoplus_\omega R \cup \bigcup_n \{\gamma_n V_n : n \in \omega\}$$

Need: If $h : f(\overline{V}) \cong f(\overline{V}')$ as left R-modules, then $\overline{V} \cong \overline{V}'$ as tagged left R-modules.

Two properties:

- Then map $h : f(\overline{V}) \cong f(\overline{V}')$ is actually an $\hat{R}$-isomorphism (uses r-purity and density of $\bigoplus_\omega R$ in $\bigoplus_\omega \hat{R}$)
- For any $\mathbf{g} \in f(\overline{V})$, $\mathbf{g} \in f(V_n)$ if and only if $\gamma_n \mathbf{g} \in f(\overline{V})$ (uses algebraic independence and r-purity).

Some corollaries to A.

A. If there is a central, non-zero divisor, non-unit $r \in R$, then Th(left R-modules) is Borel complete.

Corollary

If R is an integral domain that is not a field, then Th(R-modules) is Borel complete. Furthermore, if R is torsion-free, every free-like R-module is torsion free, hence TFAB is Borel complete.

Corollary

For any countable R, Th(left $R[x]$-modules) is Borel complete (take $r = x$). Hence Th(left R-modules (V, T) with a named $T : V \to V$) is Borel complete.

Proof: (left R-endomorphisms (V, T)) $\equiv_B$ (left $R[x]$-modules)

Corollary

For any countable R, Th(left R-modules with 4 named submodules) is Borel complete.

Proof: Fix R and let (V, T) be a left R-module with a left endomorphism $T : V \to V$.

Let $f(V, T)$ have be the left R-module with universe $V \times V$, let

- $V_0 := V \times \{0\}$;
- $V_1 := \{0\} \times V$;
- $V_2 := \{(v, v) : v \in V\}$; and
- $V_3 := \{(v, T(v)) : v \in V\}$.

Then $(V, T) \mapsto (V \times V, V_0, V_1, V_2, V_3)$ is a Borel reduction.

Thanks for listening!

Michael C. Laskowski and D. Ulrich, A proof of the Borel completeness of torsion free abelian groups, arXiv:2202.07452.