A computable functor from torsion-free abelian groups to fields

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Baer invariant

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Theorem (Baer, 37')

$G, H \in \mathfrak{TFab}_1$ are isomorphic iff there are $g \in G$ and $h \in H$ such that

$$\sum_p |\eta_p(g) - \eta_p(h)| < \infty,$$

i.e., $\eta_p(g) \neq \eta_p(h)$ for at most finitely many primes, and for each prime the difference is finite.

Example

$$\mathbb{Z} \cong \mathbb{Z}[1/2].$$

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- Let $\mathfrak{TFab}_r$ be the class of torsion-free abelian groups of rank r, with the usual abelian group language $\{0, +, -\}$.
- Independently, Kurosh ('37), Malcev ('38), and Derry ('37) found invariants for $\mathfrak{TFab}_r$.
- However, Fuchs ('73) remarked that "the theory is of minor practical value: it fails to furnish us with a useful way of deciding the isomorphy of two countable torsion-free groups".

Borel reducibility

Definition

Let E and F be two Borel equivalence relations on Polish spaces X and Y. We say E is *Borel reducible* to F and write $E \leq_B F$ if there is a Borel function $f : X \to Y$ such that for every $x_1, x_2 \in X$, $x_1 E x_2$ if and only if $f(x_1) F f(x_2)$.

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Example

Define $f_r : \mathfrak{TFab}_r \to \mathfrak{TFab}_{r+1}$ by $f_r(A) = A \oplus \mathbb{Z}$, we have

$$E_0 \sim_B \mathfrak{TFab}_1 \leq_B \mathfrak{TFab}_2 \leq_B \mathfrak{TFab}_3 \leq_B \cdots .$$

Isomorphisms on $\mathfrak{TFab}_r$

Theorem (Hjorth '99, Thomas '03)

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Theorem (Paolini–Shelah '22, Laskowski–Ulrich '22)

The space of torsion-free abelian groups with domain ω is Borel complete, i.e., the isomorphism problem is as complicated as possible.

Turing computable embedding

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Definition

Let C and D be two classes of countable structures. A *Turing computable embedding* from C to D is a computable operator Φ such that:

1. For every $A \in C$, Φ^A is (the atomic diagram of) a structure in D.
2. For $A, B \in C$, $\Phi^A \cong \Phi^B$ if and only if $A \cong B$.

We say C is *Turing reducible* to D and write $C \leq_{tc} D$ if there is a Turing computable embedding from C to D.

Groups and fields

Let $\mathfrak{T}\mathfrak{D}_r$ be the class of fields with transcendence degree r over $\mathbb{Q}$, i.e., fields isomorphic to a subfield of $\mathbb{Q}(t_1, t_2, \cdots, t_r)^{\mathrm{alg}}$, the algebraic closure of the degree r purely transcendental extension of $\mathbb{Q}$.
We work in the usual field language $\{0, 1, +, -, \cdot\}$.

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Let $\mathfrak{TD}_r$ be the class of fields with transcendence degree r over $\mathbb{Q}$, i.e., fields isomorphic to a subfield of $\mathbb{Q}(t_1, t_2, \cdots, t_r)^{\mathrm{alg}}$, the algebraic closure of the degree r purely transcendental extension of $\mathbb{Q}$.
We work in the usual field language $\{0, 1, +, -, \cdot\}$.

Theorem (H., Knight, and Miller)

$$\mathfrak{TFab}_r \leq_{tc} \mathfrak{TD}_r.$$

Theorem (HKM)

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Proof.

- Let $G \in \mathfrak{TFab}_2$ and $g, h \in G$ be a basis. Consider $\mathbb{Q}(x, y)^{\mathrm{alg}}$. We build Φ^G to be the subfield generated by $M = \{qx^a y^b \mid q \in \mathbb{Q}^{rcl}, a, b \in \mathbb{Q}, ag + bh \in G\}$ where $\mathbb{Q}^{\mathrm{rcl}}$ is the real closure of $\mathbb{Q}$. Elements in M are called *monomials*.

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- However, we do not have (uniform) access to a pair of basis. We approximate a basis by going through all pairs of elements. Whenever we see the current pair is dependent, we collapse the atomic diagram built so far into $\mathbb{Q}^{\text{rcl}}$ by letting $x = N$ and $y = M$ for sufficiently large $N, M \in \mathbb{Q}$.

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- An isomorphism $G \cong H$ induces an isomorphism $\Phi^G \cong \Phi^H$.

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- Suppose now that $\Phi^G \cong \Phi^H$, we need to find an isomorphism $G \cong H$.
- We would like to show that monomials are sent to monomials. However, this is not always true as witnessed by automorphisms of the form $x \mapsto \frac{ax+b}{cx+d}, y \mapsto y$ with $ad - bc = 1$ in $\mathbb{Q}(x, y)$.

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- If a monomial is sent to a non-monomial, then it exhibits "discrete behavior", i.e., it corresponds to a $\mathbb{Z}$ summand in G.

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- If a monomial is sent to a non-monomial, then it exhibits "discrete behavior", i.e., it corresponds to a $\mathbb{Z}$ summand in G.
- Decomposing $G = Z^k \oplus G'$, every monomial corresponding to elements in G' are sent to monomials, and we can use this to build an isomorphism $G \cong H$.

□

Computable functor

Definition

Let C and D be two categories of countable structures. A *computable functor* from C to D is a pair of computable operators Φ and Φ_* such that

1. There is a functor F from C and D.
2. For every $A \in C$, $\Phi^A = F(A)$.
3. For every $A, B \in C$ and a morphism $f : A \to B$, $\Phi_*^{D(A) \oplus f \oplus D(B)} = F(f)$.

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Theorem (HKM)

There is a computable functor from $\mathfrak{TFab}_r$ to $\mathfrak{TD}_r$ that extends the Turing computable embedding above.

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Question

Does $\mathfrak{TD}_r \leq_{tr} \mathfrak{TD}_{r+1}$? If yes, is it strict?

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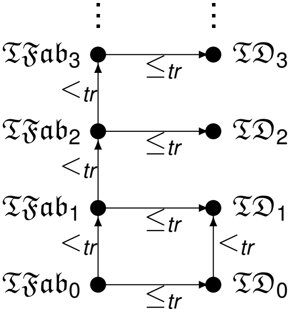
Thomas' result says we either have $\mathfrak{TD}_r \not\leq_{tr} \mathfrak{TFab}_r$ or $\mathfrak{TD}_{r+1} \not\leq_{tr} \mathfrak{TD}_r$.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- However, there are two fields $F, E \in \mathfrak{TD}_3$ such that $F \ncong E$ but $F(t_1, t_2, t_3) \cong E(t_1, t_2, t_3)$. Thus, $\Phi^F = F(t)$ does not preserve non-isomorphism for every r!

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- For strictness, we observe that the isomorphism problem on $\mathfrak{TD}_r$ is Σ_3 when $r \geq 1$, and Π_2 when $r = 0$.

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- For strictness, we observe that the isomorphism problem on $\mathfrak{TD}_r$ is Σ_3 when $r \geq 1$, and Π_2 when $r = 0$.

Question

Is there a (Borel/computable) function $\Phi : \mathfrak{TD}_r \to \mathfrak{TD}_{r+1}$ such that $E \cong F$ if and only if $\Phi(E) \cong \Phi(F)$?

Fields into groups

Question

Does $\mathfrak{T}\mathfrak{D}_r \leq_{tr} \mathfrak{T}\mathfrak{F}\mathfrak{ab}_r$*?*

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Let C and D be two categories of countable structures. A *computable* μ*-ary reduction* is a computable functional $\Phi : C^\mu \to D^\mu$ such that for every $\bar{A} \in C^\mu$, $A_\alpha \cong A_\beta$ iff $\Phi_\alpha(\bar{A}) \cong \Phi_\beta(\bar{A})$.

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In the sense of countable embedding, all $\mathfrak{TD}_r$ and $\mathfrak{TFab}_r$ collapse!

Theorem (HKM)

There is a computable countable reduction from $\mathfrak{TD}_r$ *to* $\mathfrak{TFab}_1$*.*

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- Enumerate the primes p_{ijk}. At every stage, we will make every $1 \in G_n$ divisible by p_{ijk} the same number of times, except $1 \in G_i$ is divisible by it one more time.

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- Enumerate the primes p_{ijk}. At every stage, we will make every $1 \in G_n$ divisible by p_{ijk} the same number of times, except $1 \in G_i$ is divisible by it one more time.
- At each stage, if mapping x_i to one of the first k tuples in F_j is a (partial) isomorphism on a larger domain, we make every G_m divisible by p_{ijk} one more time.

□