Effective Hausdorff Dimension and Applications II

Theodore A. Slaman

University of California Berkeley

Abstract

The Hausdorff Dimension of a set of real numbers A is a numerical indication of the geometric fullness of A. Sets of positive measure have dimension 1, but there are null sets of every possible dimension between 0 and 1.

Effective Hausdorff Dimension is a variant which incorporates computability-theoretic considerations. By work of Jack and Neil Lutz, Elvira Mayordomo, and others, there is a direct connection between the Hausdorff dimension of A and the effective Hausdorff dimensions of its elements. We will describe how this point-to-set principle works and how it allows for novel approaches to classical problems.

Hausdorff Dimension

Define a family of outer measures, parameterized by $d \in [0,1]$. For $A \subseteq 2^\omega$,

$$\mathcal{H}^d(A) = \lim_{r \to 0} \inf \left\{ \sum_i \frac{1}{2^{|\sigma_i|\, d}} : \begin{array}{l} \text{there is a cover of } A \text{ by balls} \\ B(\sigma_i) \text{ with } 1/2^{|\sigma_i|} < r \end{array} \right\}.$$

Definition

The *Hausdorff dimension* of A is as follows.

$$\begin{aligned} \dim_{\mathrm{H}}(A) &= \inf\{d \geq 0 : \mathcal{H}^d(A) = 0\} \\ &= \sup\left(\{d \geq 0 : \mathcal{H}^d(A) = \infty\} \cup \{0\}\right) \end{aligned}$$

Geometric Examples

Example

- The Cantor middle third set has dimension $\log 2/\log 3$.
- A line segment within $\mathbb{R}^2$ has dimension 1.
- Almost surely, the graph of a 2-dimensional Brownian motion has dimension 2.

Diophantine Examples

Definition

- The *exponent of irrationality* of a real number ξ is the supremum of the set of numbers z such that there are infinitely many p/q such that $0 < |\xi - p/q| < 1/q^z$.
- ξ is a *Liouville number* iff its exponent of irrationality is infinite.

Diophantine Examples

Definition

- The *exponent of irrationality* of a real number ξ is the supremum of the set of numbers z such that there are infinitely many p/q such that $0 < |\xi - p/q| < 1/q^z$.
- ξ is a *Liouville number* iff its exponent of irrationality is infinite.

Theorem (Jarník 1929 and Besicovitch 1934)

For every real number a greater than or equal to 2, the set of numbers with irrationality exponent equal to a has Hausdorff dimension $2/a$.

Baker and Schmidt (1970) generalized to approximation by algebraic numbers.

Gauge Functions and General Hausdorff Dimension

Definition

A *gauge function* is a function $h : (0, \infty) \to (0, \infty)$ which has the following properties:

- continuous
- increasing
- $\lim_{t \to 0^+} h(t) = 0$

Example

For $s > 0$, $t \mapsto t^s$ is a gauge function.

Gauge Functions and General Hausdorff Dimension

Remark

- *As above, we can associate a Hausdorff outer measure* H^h *with any gauge function* h.
- *Write* $h \prec g$ *to indicate that* $\lim_{t \to 0^+} \frac{g(t)}{h(t)} = 0$. $H^h(A) > 0$ *indicates a higher dimension than* $H^g(A) > 0$
- *A set determines a family/cut of gauge functions for which it has positive outer measure.*

Example

Let $\mathbb{L}$ denote the set of Liouville numbers.

Theorem (Olsen and Renfro (2006))

Let h be an arbitrary gauge function.

- *If $t^s \prec h$, for some $s > 0$, then $H^h(\mathbb{L}) = 0$.*
- *If $h \prec t^s$, for all $s > 0$, then $\mathbb{L}$ is not σ-finite for H^h.*

So, $\mathbb{L}$ has maximal gauge dimension among the sets of Hausdorff dimension zero.

Questions/Exercises

As far as I know, the exact gauge dimensions are not known for the following sets.

- The set of ξ with exponent of irrationality a.
- The set of ξ such that ξ has exponent of irrationality a subject to the constraint that the rationals used to approximate ξ are represented by fractions with prime denominator.

Questions/Exercises

As far as I know, the exact gauge dimensions are not known for the following sets.

- The set of ξ with exponent of irrationality a.
- The set of ξ such that ξ has exponent of irrationality a subject to the constraint that the rationals used to approximate ξ are represented by fractions with prime denominator.

Conjecture

- *Restricting to prime denominators does not change dimension with respect to the exponent of irrationality.*
- *There is a gauge function f such that if we consider intervals of diameter $f(1/q)$ around p/q and evaluate the exact dimension of the set of reals which are infinitely often f-approximable by rational numbers then restricting to rational numbers with prime denominators reduces exact dimension.*

Effective Hausdorff Dimension

formulated by measure

Introduced by Jack Lutz, this formulation due to Jan Reimann.

Effective Hausdorff Dimension

formulated by measure

Introduced by Jack Lutz, this formulation due to Jan Reimann.

Definition

- For $A \subseteq 2^\omega$, define A has *effective s-dimension Hausdorff measure* 0 iff there is a uniformly computably enumerable sequence of open sets $O_i = \bigcup_j B(\sigma_{i,j})$ such that for each i, $A \subseteq O_i$ and $\sum_j (1/2^{|\sigma_{i,j}|})^s < 1/2^i$.
- The *effective Hausdorff dimension* $\dim_{\mathrm{H}}^{\mathit{eff}}(A)$ of A is the infimum of those s such that A has effective s-dimension Hausdorff measure 0.

Effective Hausdorff Dimension

formulated by measure

Introduced by Jack Lutz, this formulation due to Jan Reimann.

Definition

- For $A \subseteq 2^\omega$, define A has *effective s-dimension Hausdorff measure* 0 iff there is a uniformly computably enumerable sequence of open sets $O_i = \bigcup_j B(\sigma_{i,j})$ such that for each i, $A \subseteq O_i$ and $\sum_j (1/2^{|\sigma_{i,j}|})^s < 1/2^i$.
- The *effective Hausdorff dimension* $\dim_{\mathrm{H}}^{\mathit{eff}}(A)$ of A is the infimum of those s such that A has effective s-dimension Hausdorff measure 0.

Remark

One can view effective Hausdorff dimension as a lightface theory for dimension just as hyperarithmetic definability is a lightface theory for Borel sets.

Effective Hausdorff Dimension

formulated by compressibility

Definition

A sequence $x \in 2^\omega$ is *algorithmically compressible by a factor of s* iff there are infinitely many ℓ such that $K(x \upharpoonright \ell) \leq s\,\ell$, where K denotes prefix-free Kolmogorov complexity.

Effective Hausdorff Dimension

formulated by compressibility

Definition

A sequence $x \in 2^\omega$ is *algorithmically compressible by a factor of s* iff there are infinitely many ℓ such that $K(x \upharpoonright \ell) \leq s\,\ell$, where K denotes prefix-free Kolmogorov complexity.

Theorem (Mayordomo 2002)

For any $x \in 2^\omega$, $\dim_H^{eff}(\{x\})$ is the infimum of the s such that x is algorithmically compressible by a factor of s.

Effective Hausdorff Dimension

formulated by compressibility

Definition

A sequence $x \in 2^\omega$ is *algorithmically compressible by a factor of s* iff there are infinitely many ℓ such that $K(x \upharpoonright \ell) \leq s\,\ell$, where K denotes prefix-free Kolmogorov complexity.

Theorem (Mayordomo 2002)

For any $x \in 2^\omega$, $\dim_{\mathrm{H}}^{\mathit{eff}}(\{x\})$ is the infimum of the s such that x is algorithmically compressible by a factor of s.

- We will abbreviate and write $\dim_{\mathrm{H}}^{\mathit{eff}}(x)$ for $\dim_{\mathrm{H}}^{\mathit{eff}}(\{x\})$.
- We can relativize to a real z and write $\dim_{\mathrm{H}}^{\mathit{eff}(z)}(x)$.

Point-to-Set for Hausdorff Dimension

Theorem (J. Lutz and N. Lutz 2017)

For $A \subseteq 2^\omega$, the Hausdorff dimension of A is equal to
the infimum over all $B \subseteq \mathbb{N}$
of the supremum over all $x \in A$
of the effective-relative-to-B Hausdorff dimension of x.

Capacitability

examples where the effective theory is helpful

Theorem

- *(Davies 1952) If A is analytic and $H^h(A) > 0$ then A has a compact subset C such that $H^h(C) > 0$.*
- *(Davies 1956 for t^s, Sion and Sjerve 1962) If A is analytic and not σ-finite for H^h then A has a compact subset C such that C is not σ-finite for H^h.*

Beyond analytic

Theorem

- *There is a set E such that E has dimension 1 and E has no perfect subset.*

Beyond analytic

Theorem

- *There is a set E such that E has dimension 1 and E has no perfect subset.*
- *$Con(ZFC)$ implies $Con(ZFC)$ and $\mathbb{R} \cap L$ has dimension 1 and has no perfect subset. Note that $\mathbb{R} \cap L$ is a Σ^1_2-set.*

Beyond analytic

Theorem

- *There is a set E such that E has dimension 1 and E has no perfect subset.*
- *$Con(ZFC)$ implies $Con(ZFC)$ and $\mathbb{R} \cap L$ has dimension 1 and has no perfect subset. Note that $\mathbb{R} \cap L$ is a Σ^1_2-set.*
- *If $V = L$ then there is a co-analytic subset E of 2^ω which is dimension 1 and has no perfect subset.*
  - *E is the maximal thin Π^1_1 set.*

Sets of Strong Dimension h

Definition

A set E has *strong dimension h* iff

$$\forall f [f \prec h \Rightarrow H^f(E) = \infty]$$

$$\forall g [h \prec g \Rightarrow H^g(E) = 0]$$

As a limiting case, E has strong dimension 0 iff for all g, $H^g(E) = 0$.

Sets of Strong Dimension h

Definition

A set E has *strong dimension h* iff

$$\forall f[f \prec h \Rightarrow H^f(E) = \infty]$$

$$\forall g[h \prec g \Rightarrow H^g(E) = 0]$$

As a limiting case, E has strong dimension 0 iff for all g, $H^g(E) = 0$.

Example

A line segment within the plane has strong dimension 1.

Sets of Strong Dimension h

Definition

A set E has *strong dimension h* iff

$$\forall f[f \prec h \Rightarrow H^f(E) = \infty]$$

$$\forall g[h \prec g \Rightarrow H^g(E) = 0]$$

As a limiting case, E has strong dimension 0 iff for all g, $H^g(E) = 0$.

Example

A line segment within the plane has strong dimension 1.

Remark

By a theorem of Besicovitch, if $H^h(A) = 0$ then there is a $j \prec h$ such that $H^j(A) = 0$. Consequently, if $H^h(A) = 0$ then A does not have strong dimension h.

Sets of Strong Dimension h

Theorem (Besicovitch 1956, generalized Rogers 1962)

If E is compact and is non-σ-finite for H^h, then there is a g such that $h \prec g$ and E is non-σ-finite for H^g.

Thus, if E is compact then E cannot have strong dimension h and be non-σ-finite for H^h.

Sets of Strong Dimension h

Theorem (Besicovitch 1956, generalized Rogers 1962)

If E is compact and is non-σ-finite for H^h, then there is a g such that $h \prec g$ and E is non-σ-finite for H^g.

Thus, if E is compact then E cannot have strong dimension h and be non-σ-finite for H^h.

Theorem (Davies 1956 for x^s, Sion and Sjerve 1962)

If E is analytic and is non-σ-finite for H^h, then there is a compact subset of E that is non-σ-finite for H^h.

Hence, we can make the above observation for analytic sets.

Sets of Strong Dimension h

Theorem (Besicovitch 1956, generalized Rogers 1962)

If E is compact and is non-σ-finite for H^h, then there is a g such that $h \prec g$ and E is non-σ-finite for H^g.

Thus, if E is compact then E cannot have strong dimension h and be non-σ-finite for H^h.

Theorem (Davies 1956 for x^s, Sion and Sjerve 1962)

If E is analytic and is non-σ-finite for H^h, then there is a compact subset of E that is non-σ-finite for H^h.

Hence, we can make the above observation for analytic sets.

Conjecture

For $a \in [0, 1)$, the set of real numbers ξ such that ξ has irrationality exponent a does not have a strong dimension.

Sets of Strong Dimension h

Now we move away from Diophantine examples.

Theorem (Besicovitch 1963)

If CH then there is a set $E \subset \mathbb{R}^2$ such that E has strong linear dimension and is non-σ-finite for linear measure.

Sets of Strong Dimension *h*

Now we move away from Diophantine examples.

Theorem (Besicovitch 1963)

If CH then there is a set $E \subset \mathbb{R}^2$ such that E has strong linear dimension and is non-σ-finite for linear measure.

Theorem (Combining Besicovitch 1963 with Erdős, Kunen and Mauldin 1981)

If $V = L$ there there is a Π^1_1 set $E \subseteq \mathbb{R}^2$ such that E has strong linear dimension and is non-σ-finite for linear measure.

Borel Conjecture

Definition

A set $E \subseteq \mathbb{R}$ has *strong measure 0* iff for any sequence of positive real numbers $\{\epsilon_i\}$ there is a sequence of open intervals $\{O_i\}$ such that for each i, O_i has length ϵ_i, and $E \subseteq \cup_{i=1}^{\infty} O_i$.

Borel Conjecture

Definition

A set $E \subseteq \mathbb{R}$ has *strong measure 0* iff for any sequence of positive real numbers $\{\epsilon_i\}$ there is a sequence of open intervals $\{O_i\}$ such that for each i, O_i has length ϵ_i, and $E \subseteq \cup_{i=1}^{\infty} O_i$.

Borel (1919) conjectured that strong measure 0 implies countable (BC).

Borel Conjecture

Definition

A set $E \subseteq \mathbb{R}$ has *strong measure 0* iff for any sequence of positive real numbers $\{\epsilon_i\}$ there is a sequence of open intervals $\{O_i\}$ such that for each i, O_i has length ϵ_i, and $E \subseteq \cup_{i=1}^{\infty} O_i$.

Borel (1919) conjectured that strong measure 0 implies countable (BC).

Theorem

- *(Sierpiński 1928) CH implies* $\neg BC$*.*
- *(Laver 1976)* $Con(ZFC)$ *implies* $Con(ZFC + BC)$*.*

Borel Conjecture

Theorem (Besicovitch 1955)

A set E has strong dimension 0 iff it has strong measure 0.

Borel Conjecture

Theorem (Besicovitch 1955)

A set E has strong dimension 0 iff it has strong measure 0.

Theorem (Another variation on Besicovitch 1963)

$\neg BC$ implies that there is a subset of $\mathbb{R}^2$ which has strong linear dimension and which is non-σ-finite for linear measure.

Two Challenges

A technical challenge:

Question

Does the Borel Conjecture imply that there do not exist h and E such that E has strong dimension h and E is not σ-finite for H^h?

Two Challenges

A technical challenge:

Question

Does the Borel Conjecture imply that there do not exist h and E such that E has strong dimension h and E is not σ-finite for H^h?

A conceptual challenge:

Question

Since the dimension of a set is not supported by its closed subsets, is there a simple characteristic sets which does support their dimension?

The End