Curve-excluding fields

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Joint work with Will Johnson (Fudan U.) and Erik Walsberg (UC Irvine).

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Throughout the talk, varieties are reduced separated schemes of finite type over some field and a curve is a variety of dimension 1. So (embedded) varieties are not required to be closed.

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An overview of model theory of fields

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- Omitting certain combinatorial configurations. (stable/NIP/simple/NSOP, etc.)
- Admitting certain strutrual description of definable sets. (strongly minimal, o-minimal, C-minimal, etc.)
- Quantifier elimination/model completeness in a reasonable language. (ACF, RCF, ACVF, p-cf, pseudofinite fields, etc.)

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Typically, the first 2 conditions follow from the last condition.

An overview of model theory of fields

Conjecture

An infinite stable field is separably closed.

Conjecture

An infinite simple field is bounded PAC.

Conjecture

An infinite NIP field is henselian.

Definition 1

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- Quantifier elimination implies model completeness. Quantifier elimination in the language of rings implies algebraically closedness.
- $\mathbb{C}, \mathbb{R}, \mathbb{Q}_p$ are all model complete. But the last 2 does not have quantifier elimination in the language of rings.
- Pseudofinite fields are model complete after a slight expansion of language of rings.

Large fields

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- $\mathbb{C}, \mathbb{R}, \mathbb{Q}_p$ are large. Any field that can be equipped with a t-Henselian topology is large.
- Any PAC field is large. Recall that PAC says that any geometrically integral variety over K has a K-point.
- In the above axiomatizations of largeness and PAC, it suffices to mention only curves.
- Empirically, all the fields that are "tame" are large.

On the other hand, there are fields that are extremely wild in the sense of model theory that are still large, e.g. $\mathbb{C}((t_1, t_2))$.

Large fields and model theory

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Several longstanding conjectures in model theory of fields have something to do with largeness as well.

Theorem 3 (Johnson, Tran, Walsberg, Y.)

The stable fields conjecture is true if the field is assumed to be large.

Theorem 4 (Johnson)

If K is the fraction field of a NIP domain R with $R \neq K$ and $\mathrm{Char}(K) = p \neq 0$, then K is large.

Definition 5

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If K is model complete, is K bounded?

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If K is model complete and infinite, is K large?

We answer the two questions negatively.

Definition 6

Let T be a theory

- T is **inductive** if it is axiomatized by $\forall\exists$-formulas. It is equivalent to that a union of chain of models is a model.
- $M \models T$ is **existentially closed** (e.c.) if any existential formula over M that holds in $N \models T$ extending M holds in M.

A crash course on model companion

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- The theory of fields is inductive. The existentially closed models are exactly the algebraically closed fields.
- Existentially closed models of formally real fields are the real closed fields.

A crash course on model companion

For an inductive theory T, we have the following:

- Any model of T embeds into an e.c. model.
- If the class of existentially closed models is an elementary class, axiomatized by T', then T' is model complete (in the language of T). In this case, we say that T has a model companion and T' is the model companion of T.

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In the previous slide: The model companion of the theory of fields is the theory of algebraically closed field; The model companion of the theory of formally real fields is the theory of real closed fields.

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Theorem 8 (Johnson, Walsberg, Y.)

T_0 has a model companion.

Corollary 9

There is a non-large model complete field.

In general, for K_0 with $\mathrm{Char}(K_0) = 0$, take any curve C_0 over K_0 of genus ≥ 2 with only finitely many (possibly 0) K_0-points.

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The theory of fields K extending K_0 (naming K_0 as constants) with the $C_0(K) = C_0(K_0)$ has a model companion $C_0\mathrm{XF}$.

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The theory of fields K extending K_0 (naming K_0 as constants) with the $C_0(K) = C_0(K_0)$ has a model companion $C_0\mathrm{XF}$.

To simplify notation, we will work with the case C_0 is the Fermat's curve with 4 $\mathbb{Q}$-points removed and $K_0 = \mathbb{Q}$.

Axioms of $C_0\mathrm{XF}$

Let T_0 denote the theory of fields K with characteristic 0 and $C_0(K) = \varnothing$.

Theorem 11

$K \models T_0$ is e.c. iff the following two conditions hold:

(1) *For any finite proper extension L/K, $C_0(L) \neq \varnothing$.*

(2) *If V is a geometrically integral variety over K, either there is a dominant $V \dashrightarrow C_0$ over K or $V(K) \neq \varnothing$.*

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Proof of $\Rightarrow$.

If K is e.c and L is a proper finite extension, then $K \not\preceq_1 L$. It means that $L \not\models T_0$. Thus (1) is satisfied.
For V a geometrically integral variety over K, if there is no dominant $V \dashrightarrow C_0$ over K, it means that $C_0(K(V)) = \varnothing$. Thus $K \preceq_1 K(V)$, so $V(K) \neq \varnothing$. Thus we have verified (2). □

Axioms of $C_0\mathrm{XF}$

Note that (2) is equivalent to (2').

(2') If V is a geometrically integral variety over K, either there is a dominant morphism $V \dashrightarrow C_0$ over K or $V(K)$ is Zariski dense in V.

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We also need the following fact.

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Suppose that V is an integral K-variety. Then $K \preceq_1 K(V)$ if and only if $V(K)$ is Zariski dense in V.

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Proof of $\Leftarrow$.

Let L/K and $L \vDash T_0$. We need to show $K \preceq_1 L$. WLOG, L is finitely generated over K. So $L = K(V)$ for some integral variety V/K. If V is not geometrically integral, then L contains a proper finite extension of K, a contradiction to (1).

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Proof of $\Leftarrow$.

Let L/K and $L \models T_0$. We need to show $K \preceq_1 L$. WLOG, L is finitely generated over K. So $L = K(V)$ for some integral variety V/K. If V is not geometrically integral, then L contains a proper finite extension of K, a contradiction to (1).
Hence V is geometrically integral. Since $L \models T_0$, it must be the case that $C_0(L) = \varnothing$. By (2'), $V(K)$ is Zariski dense in V. $\square$

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The set of morphisms between two definable set is not a definable set in general. For example, $\mathrm{Var}(\mathbb{A}^n, \mathbb{A}^1)$ is $K[X_1, \ldots, X_n]$. This can be seen as an union of definable sets indexed by $\mathbb{N}$.

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We need some bounds on the "complexity". Hopefully, here is where the assumption that the genus of C_0 is ≥ 2 comes in.

Bounding complexity

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If V is a family of curves, using Riemann-Hurwitz, we have that for any $f : V_a \dashrightarrow C_0$,

$$deg(f) \leq \frac{g(V_a) - 1}{g(C_0) - 1}.$$

This allows us to bound the complexity of a representation of f using quotients of polynomials, which makes it definable.

Bounding complexity

In general, given a normal projective varitety $X \subseteq \mathbb{P}^n$ and $f : X \dashrightarrow C_0 \subseteq \mathbb{P}^m$, using sufficiently general hyperplane intersection, we get a smooth projective curve $C \subseteq X$ and $f|_C : C \to C_0$ is a dominant morphism. Riemann-Hurwitz bounds its complexity.

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Some questions and remarks

For largeness and PAC, it suffices to mention curves in the axiomatization.

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Does (1)+ (2 for curves) characterize $C_0\mathrm{XF}$?

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And here is a meta-question.

Question 2

Do geometric invariants like "genus" carry a stability-theoretic meaning? For example: What is a model-theoretic criterion of the definability of $\mathrm{Def}(-, X)$?

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Concluding remark before we move on: $C_0\mathrm{XF}$ can be large if C_0 is a smooth projective curve and $C_0(\mathbb{Q}) = \varnothing$.

A useful lemma

Lemma 12

Let V, W be two geometrically integral varieties over K and $f : V \times W \dashrightarrow C_0$ be a rational map, then it factors over V or W generically.

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Proof.

Assume f does not factor through V. Note that there are only finitely many non-constant rational maps $W \dashrightarrow C_0$ (Mordell's conjecture for function fields), then $f(x, -)$ can be chosen to be independent of x after shrinking V. $\square$

Lemma 13

Let $K_1, K_2 \models C_0\mathrm{XF}$ and K is a common relatively algebraically closed subfield of both K_i's, then the map $\mathrm{id} : K \to K$ is a partial elementary map between K_i's.

Completions of $C_0\mathrm{XF}$

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Proof Sketch.

One can check easily that we can amalgamate K_1, K_2 over K into some $L \models T_0$ (it follows from Lemma 12). One may suppose that $L \models C_0\mathrm{XF}$. □

Completions of $C_0\mathrm{XF}$

We gather some facts needed in order to characterize the completions of $C_0\mathrm{XF}$. Recall that $\mathrm{Abs}(K) = K \cap \mathbb{Q}^{\mathrm{alg}}$.

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Corollary 14

Let $K_1, K_2 \models C_0\mathrm{XF}$, then $K_1 \equiv K_2$ iff $\mathrm{Abs}(K_1) \cong \mathrm{Abs}(K_2)$.

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With a bit of work, we characterize all the completions of $C_0\mathrm{XF}$.

Theorem 15

For any $F \models T_0$, there is a regular extension K/F such that $K \models C_0\mathrm{XF}$.
In particular, for any $F \subseteq \mathbb{Q}^{\mathrm{alg}}$ with $F \models T_0$, there is $K \models C_0\mathrm{XF}$ with $\mathrm{Abs}(K) \cong F$.

Corollary 16

The theory axiomtized by $C_0\mathrm{XF}$ *and* $\mathrm{Abs}(M) = \mathbb{Q}$ *is complete and decidable. Particularly, there is a decidable non-large field.*

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A field K is Hilbertian if for any $f \in K[X, Y]$ irreducible, there are infinitely many $a \in K$ such that $f(a, Y)$ is irreducible.

Theorem 17

C_0-exculding fields are Hilbertian.

Proof.

Take $K \vDash C_0\mathrm{XF}$. Note there is no non-constant morphism $\mathbb{A}^1 \to C_0$, so $C_0(K(t)) = \varnothing$. By Theorem 15, there is $M/K(t)$ regular such that $M \vDash C_0\mathrm{XF}$. So $K \preceq M$ by Theorem 11. So $K(t)^{\mathrm{alg}} \cap M = K(t)$. This is equivalent to Hilbertianity. □

Boundedness and finite extensions

Since Hilbertian fields are not bounded, we have

Corollary 18

Models of $C_0\mathrm{XF}$ *are not bounded. Thus, there is a unbounded model complete field.*

Boundedness and finite extensions

Similar methods enables us to study finite extensions of $C_0\mathrm{XF}$.

Corollary 19

Every proper finite extension L of $K \models C_0\mathrm{XF}$ is PAC*.*

Corollary 20 (Srinivasan)

There is a virtually large yet non-large field.

Classification-theoretic properties

Theorem 21

$C_0\mathrm{XF}$ has TP_2.

Theorem 22

$C_0\mathrm{XF}$ has NSOP_4. There is a properly NSOP_4 example.

Classification-theoretic properties

Theorem 21

$C_0\mathrm{XF}$ *has* TP_2*.*

Theorem 22

$C_0\mathrm{XF}$ *has* NSOP_4*. There is a properly* NSOP_4 *example.*

Proof.

Let $C_0 := x^4 + y^4 = -z^4$ and $K_0 = \mathbb{R}$. Let φ be saying $x - y$ is a non-zero 4-th power. It has SOP_3. $\square$

Thank you for your attention.