An analytic AKE program

DDC-2 seminar, MSRI

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- In parallel to the original theory of valued fields, we develop an extension theory in our framework.
- New is that in addition to AKE-type results for these structures, we obtain induced structure results for the coefficient field and monomial group.

Joint work with Lou van den Dries.

Outline

1. Classical AKE.
2. Denef – van den Dries' analytic expansion.
3. Some induced structure by Binyamini – Cluckers – Novikov.
4. Running the AKE program.

The AKE principle

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Theorem (Ax–Kochen–Ersov, 1965)

Let K and L be henselian valued fields of equicharacteristic 0. Then

$$K \equiv L \iff \boldsymbol{k}_K \equiv \boldsymbol{k}_L \text{ as fields, and } \Gamma_K \equiv \Gamma_L \text{ as ordered groups.}$$

AKE effects

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Let σ be any $\mathcal{L}_{\mathrm{val}}$-sentence. Then

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The AKE program runs through an extension theory of valued fields.

Gives relative elementarity, model completeness, elimination of quantifiers; and *induced structure results for lifts of the residue field and the value group.*

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Corollary

- *If $X \subseteq C_K^m$ is definable in (K, C_K, G_K), then X is even definable in the field $(C_K; 0, 1, +, -, \cdot)$.*
- *Similarly, if $Y \subseteq G_K^n$ is definable in (K, C_K, G_K), then Y is even definable in the ordered group $(G_K; 1, \cdot, \preccurlyeq)$.*

Outline

Expansion by $\mathbb{Z}[[t]]$-structure

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Can interpret the analytic structure on $\mathbb{Z}_p$ and $\mathbb{F}_p[[t]]$ through a common language.

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$$f = f(Y_1, \ldots, Y_n) = \sum_{\nu} a_\nu Y_1^{\nu_1} \cdots Y^{\nu_n}, \qquad \nu = (\nu_1, \ldots, \nu_n) \text{ ranging over } \mathbb{N}^n,$$

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Extend the language $\mathcal{L}_{\mathrm{val}}$ to $\mathcal{L}_{\mathrm{val}}^{\mathbb{Z}[[t]]}$ by augmenting an n-ary function symbol for each $f \in \mathbb{Z}[[t]]\langle Y_1, \ldots, Y_n\rangle$.

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Construe $\mathbb{Q}_p$ and $\mathbb{F}_p((t))$ as $\mathcal{L}_{\mathrm{val}}^{\mathbb{Z}[[t]]}$-structures.
$f \in \mathbb{Z}[[t]]\langle Y\rangle$ only takes values in $\mathbb{Z}_p$ and $\mathbb{F}_p[[t]]$.

An analytic AKE principle

Theorem (van den Dries, 1992)

Let σ *be any* $\mathcal{L}_{\mathrm{val}}^{\mathbb{Z}[[t]]}$*-sentence. Then*

$$\mathbb{Q}_p \models \sigma \iff \mathbb{F}_p((t)) \models \sigma$$

for all but finitely many primes p.

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Gives relative elementarity, model completeness, elimination of quantifiers, but **not** induced structure results for the coefficient field and monomial group.

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We consider valuation rings with A-analytic structure and their fraction fields as $\mathcal{L}_{\mathrm{val}}^{A}$-structures.

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In connection with a non-archimedean analogue of Pila-Wilkie type counting result, BCN consider a 3-sorted structure $\mathcal{M}$ comprised of:

the analytic valued field $\mathbb{C}((t))_{\mathrm{an}}$, the field $\mathbb{C}$, the ordered abelian group $\mathbb{Z}$.

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Proposition (Binyamini – Cluckers – Novikov, 2022)

If $P \subseteq \mathbb{C}((t))^n$ is definable in $\mathcal{M}$, then $P \cap \mathbb{C}^n$ is definable in the field $(\mathbb{C}; 0, 1, +, -, \cdot)$.

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Lou's "analytic AKE" results do give that any subset of $\mathbb{C}^n$ definable in $\mathcal{M}$ is definable in the field $\mathbb{C}$, but that's not enough.

Sample induced structure results

We consider the 1-sorted structure $\left(C((t))_{\mathrm{an}}, C, t^{\mathbb{Z}}\right)$, which includes lifts for the residue field and value group.

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Outline

1. Classical AKE.
2. Denef – van den Dries' analytic expansion.
3. Some induced structure by Binyamini – Cluckers – Novikov.
4. Running the AKE program.

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Our assumptions give that viable valuation A-rings have:

piecewise uniform Weierstrass division with respect to parameters.

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$R\langle a \rangle \subseteq R\langle u_\rho \rangle \subseteq R_a$, and we discover that $R_a = \bigcup_{\rho > \rho_0} R\langle u_\rho \rangle$.

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- Is K_a always an immediate extension of $K(a)$?

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