Morse theory on moduli spaces of pairs and the Bogomolov–Miyaoka–Yau inequality

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Outline

Collaborators and references

This talk is based on the following monographs [19, 21]:

- *Virtual Morse–Bott index, moduli spaces of pairs, and applications to topology of smooth four-manifolds* (with **Tom Leness**), Memoirs of the American Mathematical Society, in press, xiv+274 pages, arXiv:2010.15789
- *Białnicki–Birula theory, Morse–Bott theory, and resolution of singularities for analytic spaces*, xii+189 pages, arXiv:2206.14710

Work in progress on a special case of our program for smooth complex, projective algebraic surfaces is joint with Tom Leness and **Richard Wentworth**.

Bogomolov–Miyaoka–Yau inequality: theorems and conjectures

We begin by recalling the

Theorem 1.1 (Bogomolov–Miyaoka–Yau inequality for complex surfaces of general type)

(See Miyaoka [43, Theorem 4] and Yau [61, Theorem 4].) If X is a compact, complex surface of general type, then

$$c_1(X)^2 \leq 3c_2(X). \tag{1}$$

Here, $c_1(X)$ and $c_2(X)$ are the Chern classes of the holomorphic tangent bundle, $\mathscr{T}_X \cong T^{0,1}X$.

In [43], Miyaoka proved Theorem 1.1 using algebraic geometry.

See Barth, Hulek, Peters, and Van de Ven [8, Section VII.4] for a simplification of Miyaoka's proof of Theorem 1.1.

Bogomolov [10] proved a weaker version of (1), namely $c_1(X)^2 \leq 4c_2(X)$.

Overview of Yau's proof of Theorem 1.1

Yau proved (1) in a slightly more restricted setting than Theorem 1.1 as a consequence of his proof of the Calabi Conjectures.

Let X be a complex manifold of dimension $n \geq 2$ with Kähler metric g.

Let ω_g be the closed, real $(1,1)$-form and $\mathrm{Ric}(g)$ be the Ricci curvature 2-form associated to g.

One calls g a *Kähler–Einstein metric* if there exists a $\lambda \in \mathbb{R}$ such that

$$\mathrm{Ric}(g) = \lambda \omega_g. \tag{2}$$

As part of his work on the Calabi Conjectures, Yau proved the

Theorem 1.2 (Yau)

(See Yau [62, Theorems 1 and 2, pp. 363–364] and [61, Theorem 1], or Barth, Hulek, Peters, and Van de Ven [8, Theorem 15.2, p. 52].) Let X be a compact, complex manifold of dimension $n \geq 2$ such that $-c_1(X)$ can be represented by a Kähler form. Then X admits a Kähler–Einstein metric.

The Kähler–Einstein metric g produced by Theorem 1.2 obeys (2) with constant $\lambda < 0$.

Lemma 1.3 (Chern–Weil inequality)

(See Tosatti [55, Lemma 2.6] for an exposition.) If (X, ω) is a compact Kähler–Einstein manifold of dimension $n \geq 2$, then

$$\left(\frac{2(n+1)}{n} c_2(X) - c_1(X)^2\right) \cdot [\omega]^{n-2} \geq 0, \tag{3}$$

with equality if and only if ω has constant holomorphic sectional curvature.

By applying Theorem 1.2 and Lemma 1.3, Yau proved

Theorem 1.4 (Bogomolov–Miyaoka–Yau inequality and uniformization)

(See Yau [61, Theorem 4].) Let X be a compact, complex Kähler surface with ample canonical bundle. Then inequality (1) *holds and equality occurs if and only if X is covered biholomorphically by the ball in $\mathbb{C}^2$.*

Recall that a holomorphic line bundle L over a complex manifold X is defined to be *ample* if there exists a holomorphic embedding $f : X \to \mathbb{CP}^N$ such that $L^{\otimes k} \cong f^*\mathscr{O}_{\mathbb{CP}^N}(1)$ for some positive integers k and N.

A well-known theorem due to Kodaira (see Angella and Spotti [4, Theorem 2.5, p. 207]) asserts that a holomorphic line bundle over a compact complex manifold admits a positive Hermitian metric if and only if it is ample.

If X is *Fano*, so $c_1(X) > 0$, that is the first Chern class can be represented by a Kähler form, Kodaira's theorem says that this condition is equivalent to the anti-canonical line bundle being ample.

If $c_1(X) < 0$, Kodaira's theorem says that this condition is equivalent to the canonical line bundle being ample and, in particular, X is a *surface of of general type*.

Lastly, we recall that Simpson [45, p. 871] proved Theorem 1.1 as a corollary of his main theorem [45, p. 870] on existence of Hermitian–Einstein (or Hermitian–Yang–Mills) connection on a stable Higgs bundle of rank 3 over X and the following

Theorem 1.5 (Bogomolov–Gieseker inequality)

(See Kobayashi [38, Theorem 4.4.7] or Lübke and Teleman [42, Corollary 2.2.4].) Let (E, h) be a Hermitian vector bundle over of rank r over a compact, complex Kähler manifold of dimension $n \geq 2$. If (E, h) admits a Hermitian–Einstein connection, then

$$\int_X \left(2rc_2(E) - (r-1)c_1(E)^2\right) \wedge \omega^{n-2} \geq 0. \tag{4}$$

According to Bogomolov [10] and Gieseker [28], inequality (4) holds for any holomorphic, slope semi-stable vector bundle over a smooth compact, complex projective surface (see Huybrechts and Lehn [37, Theorem 3.4.1, p. 80]).

For a closed topological four-manifold X, we define

$$c_1(X)^2 := 2e(X) + 3\sigma(X) \quad \text{and} \quad \chi_h(X) := \frac{1}{4}(e(X) + \sigma(X)),$$

where $e(X) = 2 - 2b_1(X) + b_2(X)$ and $\sigma(X) = b^+(X) - b^-(X)$ are the *Euler characteristic* and *signature* of X, respectively.

If Q_X is the intersection form on $H_2(X;\mathbb{Z})$, then $b^\pm(X)$ are the dimensions of the maximal positive and negative subspaces of Q_X on $H_2(X;\mathbb{R})$.

We call X **standard** if it is closed, connected, oriented, and smooth with odd $b^+(X) \geq 3$ and $b_1(X) = 0$.

Conjecture 1 (Bogomolov–Miyaoka–Yau (BMY) inequality for four-manifolds with non-zero Seiberg–Witten invariants)

If X is a standard four-manifold of Seiberg–Witten simple type with a non-zero Seiberg–Witten invariant, then

$$c_1(X)^2 \leq 9\chi_h(X). \tag{5}$$

If X obeys the hypotheses of Conjecture 1, then it has an almost complex structure J and for inequality (5), which is equivalent to (1), namely

$$c_1(X)^2 \leq 3c_2(X),$$

where the Chern classes are those of $T^{0,1}X$.

Conjecture 1 is based on Stern [46, Problem 4] (see also Kollár [39]), but often stated for simply connected *symplectic* four-manifolds — see Gompf and Stipsicz [31, Remark 10.2.16 (c)] or Stern [46, Problem 2].

Taubes [51, 52] showed that symplectic four-manifolds have Seiberg–Witten simple type with non-zero Seiberg–Witten invariants.

Szabó [49] proved existence of four-dimensional, *non-symplectic*, smooth manifolds with non-zero Seiberg–Witten invariants.

Conjecture 1 has inspired contructions by topologists of examples to shed light on inequality (5), including work of Akhmedov, Hughes, and Park [1, 2, 3], Baldridge, Kirk, and Li [5, 6, 7], Bryan, Donagi, and Stipsicz [12], Fintushel and Stern [24], Gompf and Mrowka [29, 30], Hamenstädt [32], Park and Stipsicz [44, 47, 48], I. Smith, Torres [54], and others.

Conjecture 1 holds for all examples that satisfy the hypotheses, but the BMY inequality (5) can fail for four-manifolds with zero Seiberg–Witten invariants, such as a connected sum of two or more copies of $\mathbb{CP}^2$.

LeBrun [40] proved the BMY inequality (5) for Einstein four-manifolds with non-zero Seiberg–Witten invariants.

Conjecture 2 (Existence of ASD connections with small instanton number)

Assume the hypotheses of Conjecture 1 and let E be a complex rank two, Hermitian vector bundle over X whose associated SO(3) bundle $\mathfrak{su}(E)$ has first Pontrjagin number obeying the basic lower bound,

$$p_1(\mathfrak{su}(E)) \geq c_1(X)^2 - 12\chi_h(X). \tag{6}$$

Let g be a Riemannian metric on X that is generic in the sense of Freed and Uhlenbeck [16, 26]. Then there exists a smooth, projectively g-anti-self-dual Yang–Mills unitary connection A on E, so the curvature $F_A \in \Omega^2(\mathfrak{u}(E))$ obeys

$$(F_A^+)_0 = 0 \in \Omega^+(X; \mathfrak{su}(E)), \tag{7}$$

where $^+ : \wedge^2(T^*X) \to \wedge^+(T^*X)$ and $(\,\cdot\,)_0 : \mathfrak{u}(E) \to \mathfrak{su}(E)$ are orthogonal projections.

One has $c_1(X)^2 - 12\chi_h(X) = -e(X) = -c_2(X)$ by [31, Section 1.4.1], so (6) $\iff$ the *instanton number* obeys $\kappa := -\frac{1}{4}p_1(\mathfrak{su}(E)) \leq \frac{1}{4}e(X)$.

We now explain why Conjecture 2 $\Longrightarrow$ Conjecture 1:

For $w \in H^2(X;\mathbb{Z})$ and $4\kappa \in \mathbb{Z}$, let (E,h) be a rank-2 Hermitian bundle over X with $c_1(E) = w$, fixed unitary connection A_d on $\det E$, and

$$p_1(\mathfrak{su}(E)) = c_1(E)^2 - 4c_2(E) = -4\kappa.$$

The moduli space of projectively anti-self-dual (ASD) connections on E is

$$M_\kappa^w(X,g) := \{A : (F_A^+)_0 = 0\}/\mathscr{G}_E.$$

$\mathscr{G}_E$ is the group of determinant-one, unitary automorphisms of (E,h).

The expected dimension of $M_\kappa^w(X,g)$ is given by [16]

$$\operatorname{exp}\dim M_\kappa^w(X,g) = -2p_1(\mathfrak{su}(E)) - 6\chi_h(X). \tag{8}$$

When g is *generic* in the sense of [16, 26], then $M_\kappa^w(X,g)$ is a smooth (usually non-compact) manifold if non-empty.

If Conjecture 2 holds, then $\mathfrak{su}(E)$ admits a g-anti-self-dual connection when the basic lower bound (6) holds and the metric g on X is generic.

The moduli space $M_\kappa^w(X, g)$ is thus a non-empty, smooth manifold and so

$$\exp\dim M_\kappa^w(X, g) \geq 0.$$

This yields the Bogomolov–Miyaoka–Yau inequality (5) since

$$\begin{aligned} 0 &\leq \frac{1}{2}\exp\dim M_\kappa^w(X, g) \\ &= -p_1(\mathfrak{su}(E)) - 3\chi_h(X) \quad \text{(by (8))} \\ &\leq -\left(c_1(X)^2 - 12\chi_h(X)\right) - 3\chi_h(X) \quad \text{(by (6))} \\ &= -c_1(X)^2 + 9\chi_h(X). \end{aligned}$$

Taubes [50] proved existence of solutions to the ASD equation (7) only when the instanton number $\kappa(E) = -\frac{1}{4}p_1(\mathfrak{su}(E))$ is sufficiently large.

The difficulty in proving Conjecture 2 is because the basic lower bound (6) implies that $\kappa(E)$ is small and Taubes' gluing method does not apply.

We aim to prove Conjecture 2 via existence of projectively anti-self-dual connections as absolute minima of a Hamiltonian function f for the circle action on the moduli space of non-Abelian monopoles.

Non-Abelian monopoles and the Bogomolov–Miyaoka–Yau inequality

Let (ρ, W) be a *spin^c structure* and (E, h) be a Hermitian vector bundle over an oriented, Riemannian four-manifold (X, g).

Consider the affine space of unitary connections A on E that induce a fixed unitary connection A_d on $\det E$ and sections Φ of $W^+ \otimes E$.

We call (A, Φ) a non-Abelian monopole if

$$\begin{aligned} (F_A^+)_0 - \rho^{-1}(\Phi \otimes \Phi^*)_{00} &= 0, \\ D_A \Phi &= 0, \end{aligned} \tag{9}$$

where the section $(\Phi \otimes \Phi^*)_{00}$ of $\mathfrak{su}(W^+) \otimes \mathfrak{su}(E)$ is the trace-free component of $\Phi \otimes \Phi^*$ of $\mathfrak{u}(W^+) \otimes \mathfrak{u}(E)$ and D_A is the Dirac operator and $\rho : \wedge^+(T^*X) \to \mathfrak{su}(W^+)$ is an isomorphism of SO(3) bundles.

The moduli space of non-Abelian monopoles is

$$\mathscr{M}_{\mathfrak{t}} := \{(A, \Phi) \text{ obeying } (9)\} / \mathscr{G}_E.$$

The space $\mathscr{M}_{\mathfrak{t}}$ has a decomposition as a disjoint union of subsets

$$\mathscr{M}_{\mathfrak{t}} = \mathscr{M}_{\mathfrak{t}}^{*,0} \sqcup \mathscr{M}_{\mathfrak{t}}^{\{\Phi\equiv 0\}} \sqcup \mathscr{M}_{\mathfrak{t}}^{\{A \text{ reducible}\}},$$

where $\mathscr{M}_{\mathfrak{t}}^{*,0} \subset \mathscr{M}_{\mathfrak{t}}$ is the subspace of *irreducible, non-zero-section pairs*, a *finite-dimensional smooth manifold* for generic geometric perturbations (see F. and Leness [22, 20] and Teleman [53]).

Our hypothesis in Conjecture 1 that X has a non-zero Seiberg–Witten invariant ensures that the subspace $\mathscr{M}_{\mathfrak{t}}^{*,0}$ is non-empty.

Multiplication by $\mathbb{C}^*$ on sections Φ induces an S^1 action on $\mathscr{M}_\mathfrak{t}$ with two types of fixed points, represented by pairs (A, Φ) such that

- $\Phi \equiv 0$, or
- A is a *reducible connection* for some splitting, $E = L_1 \oplus L_2$.

For points $[A, \Phi] \in \mathscr{M}_\mathfrak{t}$, there are bijections between

- the subset of $\mathscr{M}_\mathfrak{t}^{\{\Phi\equiv 0\}}$, where $\Phi \equiv 0$, and the moduli space $M_\kappa^w(X, g)$ of anti-self-dual connections, and
- subsets of $\mathscr{M}_\mathfrak{t}^{\{A \text{ reducible}\}}$, where A is reducible for a splitting $E = L_1 \oplus L_2$, and a moduli space $M_\mathfrak{s}$ of Seiberg–Witten monopoles defined by a spin^c structure $\mathfrak{s} = (\rho, W \otimes L_1)$.

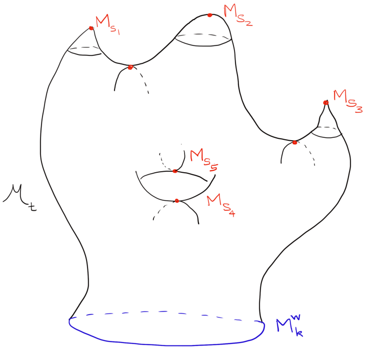

Figure 2.1: Non-Abelian monopole moduli space $\mathscr{M}_{\mathfrak{t}}$ with Seiberg–Witten moduli subspaces $\sqcup_i M_{\mathfrak{s}_i} \cong \mathscr{M}_{\mathfrak{t}}^{\{A \text{ reducible}\}}$ and moduli subspace $M_\kappa^w(X, g) \cong \mathscr{M}_{\mathfrak{t}}^{\{\Phi \equiv 0\}}$ of anti-self-dual connections

We use an extension of Morse-Bott theory from smooth manifolds to analytic spaces to try to prove Conjecture 2 on existence of anti-self-dual connections on $\mathfrak{su}(E)$.

To motivate our version of Morse-Bott theory, we describe an idealized model case. Hitchin's Hamiltonian function,

$$f : \mathscr{M}_{\mathfrak{t}} \ni [A, \Phi] \mapsto f[A, \Phi] := \frac{1}{2}\|\Phi\|^2_{L^2(X)} \in \mathbb{R}, \tag{10}$$

is continuous and smooth on smooth strata of $\mathscr{M}_{\mathfrak{t}}$ and attains its absolute minimum value of zero on the moduli subspace $M^w_\kappa(X, g)$, *if non-empty*.

(Hitchin used Morse-Bott theory for f in (10) in his analysis [34] of the topology of the moduli space of Higgs pairs over a Riemann surface.)

We temporarily assume that $\mathscr{M}_{\mathfrak{t}}$ is a smooth manifold (usually false), in which case f is also smooth, and that $\mathscr{M}_{\mathfrak{t}}$ is compact (usually false).

The moduli space $\mathscr{M}_{\mathfrak{t}}$ is equipped with the L^2 Riemannian metric.

Assume further that f is *Morse–Bott* on $\mathscr{M}_{\mathfrak{t}}$ and that its *critical submanifolds* comprise the moduli subspace $M_\kappa^w(X, g)$ of *anti-self-dual connections* (if non-empty) and the moduli subspaces $M_{\mathfrak{s}_i}$ of *Seiberg–Witten monopoles*.

Because f is Morse–Bott on $\mathscr{M}_{\mathfrak{t}}$, if $[A, \Phi] \in \mathscr{M}_{\mathfrak{t}}$ is a critical point, so

$$\operatorname{Ker} df[A, \Phi] = T_{[A,\Phi]}\mathscr{M}_{\mathfrak{t}},$$

then the Hessian of f (defined by the L^2 metric) obeys

$$\operatorname{Ker} \operatorname{Hess} f[A, \Phi] = T_{[A,\Phi]} \operatorname{Crit} f,$$

and the tangent space $T_{[A,\Phi]}\mathscr{M}_{\mathfrak{t}}$ has an orthogonal splitting,

$$T_{[A,\Phi]}\mathscr{M}_{\mathfrak{t}} = T^+_{[A,\Phi]}\mathscr{M}_{\mathfrak{t}} \oplus T^-_{[A,\Phi]}\mathscr{M}_{\mathfrak{t}} \oplus T^0_{[A,\Phi]}\mathscr{M}_{\mathfrak{t}}.$$

The subspaces $T^{\pm}_{[A,\Phi]}\mathscr{M}_{\mathfrak{t}}$ where $\operatorname{Hess} f[A,\Phi]$ is *positive* or *negative definite* are tangent spaces to the stable and unstable manifolds through $[A,\Phi]$.

The subspace $T^{0}_{[A,\Phi]}\mathscr{M}_{\mathfrak{t}}$ where $\operatorname{Hess} f[A,\Phi]$ is zero is the *tangent space to the critical submanifold* $\operatorname{Crit} f$.

The Morse–Bott signature of the critical point $[A,\Phi]$ is given by

$$\lambda^{+}_{[A,\Phi]}(f) := \dim T^{\pm}_{[A,\Phi]}\mathscr{M}_{\mathfrak{t}} \quad \text{and} \quad \lambda^{0}_{[A,\Phi]}(f) := \dim T^{0}_{[A,\Phi]}\mathscr{M}_{\mathfrak{t}},$$

comprising the Morse–Bott index, co-index, and nullity.

Observation 2.1 (Positive Morse–Bott indices for Seiberg–Witten critical points $\implies$ existence of anti-self-dual connections)

If the Morse–Bott index of every Seiberg–Witten critical submanifold is positive, then the critical submanifold given by the moduli space of anti-self-dual connections is non-empty.

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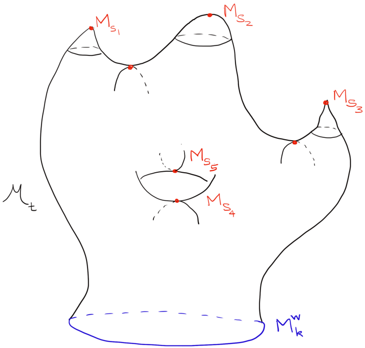

Figure 2.2: Non-Abelian monopole moduli space $\mathscr{M}_\mathfrak{t}$ with Seiberg–Witten moduli subspaces $M_{\mathfrak{s}_i}$ and moduli subspace $M_\kappa^w(X, g)$ of anti-self-dual connections

One can compute the Morse–Bott index of a critical point using Frankel's Theorem [25], used by Hitchin [34] for the moduli space of Higgs monopoles on a Hermitian bundle (E, h), whose rank and degree are coprime, over a Riemann surface.

Suppose that (M, g, J) is an almost Hermitian manifold that admits a smooth circle action $\rho : S^1 \times M \to M$ and a circle-invariant, non-degenerate two-form, $\omega = g(\cdot, J\cdot)$.

Let f be a Hamiltonian function for the circle action, so

$$df = \iota_\xi \omega,$$

where the smooth vector field ξ on M is the generator of the S^1 action.

Theorem 2.2 (Frankel's theorem for almost Hermitian manifolds)

(See Frankel [25, Section 3] for complex, Kähler manifolds and [21, Theorem 2] for almost Hermitian manifolds.)

1. *A point $p \in M$ is a critical point of $f \iff p$ is a fixed point of the circle action ρ on M.*
2. *The Hamiltonian, f, is Morse–Bott at each critical point p, with Morse–Bott signature $(\lambda_p^+(f), \lambda_p^-(f), \lambda_p^0(f))$ given by the dimensions $(\lambda_p^+(\rho), \lambda_p^-(\rho), \lambda_p^0(\rho))$ of the positive, negative, and zero weight spaces for the circle action ρ_* on the tangent space T_pM.*

If X is a compact, complex Kähler surface, then the subspace $\mathscr{M}_{\mathfrak{t}}^{\mathrm{sm}}$ of smooth points is a complex Kähler manifold with circle-invariant Kähler form $\boldsymbol{\omega}$ and f in (10) is a Hamiltonian function for this circle action.

Thus, if X is Kähler, then the following are equivalent for $[A, \Phi] \in \mathscr{M}_{\mathfrak{t}}^{\mathrm{sm}}$:

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- $[A, \Phi]$ is a critical point of f,
- $[A, \Phi]$ is a fixed point of the circle action on $\mathscr{M}_{\mathfrak{t}}^{\mathrm{sm}}$,
- A is reducible, so (A, Φ) is a Seiberg–Witten monopole, or $\Phi \equiv 0$ and A is projectively anti-self-dual.

The preceding ideas extend to the case of fixed points $[A, \Phi] \in \mathscr{M}_{\mathfrak{t}}$ that are *not* necessarily smooth points.

In [21], we apply the *Hirzebruch–Riemann–Roch Theorem* to compute a *virtual Morse–Bott signature* for each fixed point $[A, \Phi] \in \mathscr{M}_{\mathfrak{t}}$ represented by a Seiberg–Witten monopole and show that its virtual Morse–Bott index is positive and thus *cannot be a local minimum*.

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Few of our assumptions for the idealized model hold in practice:

1. Singular critical points. The moduli subspace $M_\kappa^w(X)$ of anti-self-dual connections and moduli subspaces $M_{\mathfrak{s}_i}$ of Seiberg–Witten monopoles are *singularities* in the moduli space $\mathscr{M}_\mathfrak{t}$ of non-Abelian monopoles (even when those subspaces are smooth manifolds).
2. Non-compact. The moduli space $\mathscr{M}_\mathfrak{t}$ of non-Abelian monopoles is non-compact due to *Uhlenbeck energy bubbling* [56, 57].
3. Non-Kähler. The moduli space $\mathscr{M}_\mathfrak{t}$ of non-Abelian monopoles is *not necessarily a complex Kähler manifold* (away from singularities) when the almost complex structure J on X is not assumed integrable and the fundamental two-form $\omega = g(\cdot, J\cdot)$ is not assumed closed.

The non-compactness of $\mathscr{M}_{\mathfrak{t}}$ can be addressed in two ways:

When X is a smooth, complex projective surface, then the Hitchin–Kobayashi correspondence gives an isomorphism of real analytic spaces between $\mathscr{M}_{\mathfrak{t}}$ and the moduli space $\mathfrak{M}^{\mu}(E)$ of slope stable, holomorphic pairs.

The moduli space $\mathfrak{M}^{\mu}(E)$ admits a Gieseker compactification as a moduli space $\mathfrak{M}^{ss}(E)$ of *pairs* of coherent sheaves that are semistable in the sense of Gieseker–Maruyama, together with sections (see Dowker [17], Huybrechts and Lehn [36, 35], Lin [41], and Wandel [58]).

When X is a smooth Riemannian four-manifold, then $\mathscr{M}_{\mathfrak{t}}$ admits an Uhlenbeck (or bubble-tree) compactification $\bar{\mathscr{M}}_{\mathfrak{t}}$ given by the Uhlenbeck closure of $\mathscr{M}_{\mathfrak{t}}$ in the space of *ideal non-Abelian monopoles*,

$$\mathscr{I}\mathscr{M}_{\mathfrak{t}} := \bigsqcup_{\ell=0}^{\infty} \left(\mathscr{M}_{\mathfrak{t}(\ell)} \times \mathrm{Sym}^{\ell}(X)\right), \tag{11}$$

where $\mathfrak{t}(\ell) = (\rho, W \otimes E_\ell)$ and (E_ℓ, h_ℓ) is a rank-2 Hermitian vector bundle over X with fixed unitary connection A_d on $\det E_\ell \cong \det E$ and

$$c_1(E_\ell) = c_1(E), \quad c_2(E_\ell) = c_2(E) - \ell, \quad p_1(\mathfrak{su}(E_\ell)) = p_1(\mathfrak{su}(E)) + 4\ell.$$

We call the intersection of $\bar{\mathscr{M}}_{\mathfrak{t}}$ with $\mathscr{M}_{\mathfrak{t}(\ell)} \times \mathrm{Sym}^\ell(X)$ its ℓ-th level.

Either choice of compactification (Gieseker or Uhlenbeck) introduces more singularities and leads back to the first difficulty that the moduli space $\mathscr{M}_{\mathfrak{t}}$ of non-Abelian monopoles (and any compactification) has singularities.

The moduli space $\mathscr{M}_{\mathfrak{t}}$ of non-Abelian monopoles is an *analytic space*.

Thus, *resolution of singularities* (in the sense of Hironaka [33]) can be used to partially extend some aspects of classical Morse–Bott theory from smooth manifolds to real or complex analytic spaces.

We summarize our program to prove Conjecture 1:

1. Prove existence of feasible spin^u structure $\mathfrak{t} = (\rho, W \otimes E)$ with
   - $p_1(\mathfrak{su}(E))$ obeying the basic lower bound (6);
   - Moduli subspace $\mathscr{M}_\mathfrak{t}^{*,0}$ of irreducible, non-zero-section non-Abelian monopoles is non-empty.
2. Prove that all critical points of Hitchin's function on $\bar{\mathscr{M}}_\mathfrak{t}$ are
   - points in the anti-self-dual moduli subspace $M_\kappa^w(X, g) \subset \mathscr{M}_\mathfrak{t}$; or
   - points in moduli subspaces $M_\mathfrak{s} \subset \bar{\mathscr{M}}_\mathfrak{t}$ of Seiberg–Witten monopoles.
3. Prove that all points in moduli subspaces $M_\mathfrak{s} \subset \bar{\mathscr{M}}_\mathfrak{t}$ of Seiberg–Witten monopoles have positive virtual Morse–Bott index.

The above three steps in our program are completed in our monograph [21] for $\mathscr{M}_\mathfrak{t}$ when X is Kähler, but *not* for $\bar{\mathscr{M}}_\mathfrak{t}$ or $\mathfrak{M}^{ss}(E)$ or X smooth.

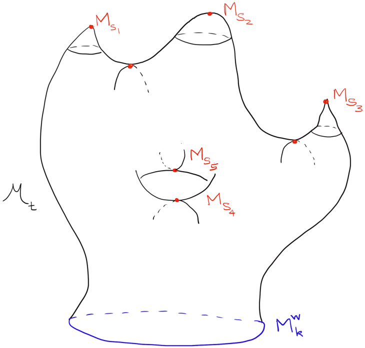

Figure 2.3: SO(3) monopole moduli space $\mathscr{M}_{\mathfrak{t}}$ with Seiberg–Witten moduli subspaces $M_{\mathfrak{s}_i}$ and moduli subspace $M_{\kappa}^{w}(X,g)$ of anti-self-dual connections

Virtual Morse–Bott indices for Hamiltonian functions of circle actions on complex analytic subspaces of complex, Kähler manifolds with holomorphic $\mathbb{C}^*$ actions

Inspired by Hitchin [34], we extend the definition of the index of a Morse–Bott function at a critical point in a smooth manifold to the case of

> *A critical point of a Hamiltonian function for the circle action on $\mathbb{C}^*$-invariant, closed, complex analytic subspace of a complex, Kähler manifold with a holomorphic $\mathbb{C}^*$ action.*

Complex analytic spaces with circle actions are pervasive in gauge theory over complex Kähler manifolds or smooth complex, projective varieties:

- Moduli spaces of Higgs bundles (Hitchin–Simpson pairs),
- Moduli spaces of projective vortices (Bradlow pairs),
- Moduli spaces of non-Abelian monopoles,
- Moduli spaces of stable pairs of holomorphic bundles and sections.

Białynicki–Birula theory for holomorphic $\mathbb{C}^*$ actions on complex manifolds

Based on results due to Białynicki–Birula [9] for torus actions on smooth algebraic varieties (also Carrell and Sommese [13], Fujiki [27]), we have

Definition 3 (Białynicki–Birula decompositions for complex manifolds)

Let X be a complex manifold and $\mathbb{C}^* \times X \to X$ be a holomorphic $\mathbb{C}^*$ action such that the subset $X^0 := X^{\mathbb{C}^*} \subset X$ of fixed points of the $\mathbb{C}^*$ action is non-empty with at most countably many connected components, X^0_α for $\alpha \in \mathscr{A}$, that are embedded complex submanifolds of X. For each $\alpha \in \mathscr{A}$, define

$$X^+_\alpha := \left\{ z : \lim_{\lambda \to 0} \lambda \cdot z \in X^0_\alpha \right\} \quad \text{and} \quad X^-_\alpha := \left\{ z : \lim_{\lambda \to \infty} \lambda \cdot z \in X^0_\alpha \right\}, \tag{12}$$

so the subsets $X^+_\alpha \subset X$ are $\mathbb{C}^*$-invariant and mutually disjoint for all $\alpha \in \mathscr{A}$ and similarly for the the subsets $X^-_\alpha \subset X$ for all $\alpha \in \mathscr{A}$, and

$$\pi^+_\alpha(z) := \lim_{\lambda \to 0} \lambda \cdot z, \quad \text{for all } z \in X^+_\alpha, \quad \text{and} \quad \pi^-_\alpha(z) := \lim_{\lambda \to \infty} \lambda \cdot z, \quad \text{for all } z \in X^-_\alpha. \tag{13}$$

Definition 3 (Białynicki–Birula decompositions for complex manifolds)

Then X has a *(mixed, plus, or minus) Białynicki–Birula decomposition* if the following hold:

1. Each X_α^+ is an embedded complex submanifold of X;
2. The natural map $\pi_\alpha^+ : X_\alpha^+ \to X_\alpha^0$ is a $\mathbb{C}^*$-equivariant, holomorphic, maximal-rank surjection;

and the analogous properties hold for the subsets X_α^- and for the maps $\pi_\alpha^- : X_\alpha^- \to X_\alpha^0$. Furthermore, we require that:

3. The normal bundles $N_{X_\alpha^0/X_\alpha^+}$ of X_α^0 in X_α^+ and $N_{X_\alpha^0/X_\alpha^-}$ of X_α^0 in X_α^- are subbundles of the normal bundle $N_{X_\alpha^0/X}$ of X_α^0 in X. There is a weight-sign decomposition defined by the S^1 action on X induced by the $\mathbb{C}^*$ action,

$$TX \restriction X_\alpha^0 = T^0 X_\alpha \oplus N^+_{X_\alpha^0/X} \oplus N^-_{X_\alpha^0/X}; \tag{14}$$

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Definition 3 (Białynicki–Birula decompositions for complex manifolds)

4. For some $A \subset \mathscr{A} \times \{+,-\}$, the space X is expressed as a disjoint union,

$$X = \bigsqcup_{(\alpha,j)\in A} X_\alpha^j. \tag{15}$$

If one can express X as

$$X = \bigsqcup_{\alpha\in\mathscr{A}} X_\alpha^+ \quad \text{or} \quad X = \bigsqcup_{\alpha\in\mathscr{A}} X_\alpha^-, \tag{16}$$

where the union is disjoint, then we say that X has a *plus* or *minus decomposition*, respectively and, otherwise, if X is expressed as in (15), that it has a *mixed decomposition*.

One has the following result due to Białynicki–Birula [9], Carrell and Sommese [13, 14, 15], Fujiki [27], and Yang [60] (see F. [19, Theorem 3] for a generalization).

Theorem 4 (Białynicki–Birula decomposition for Kähler manifolds)

If X is a compact, complex Kähler manifold with a holomorphic action $\mathbb{C}^* \times X \to X$, then it admits plus and minus Białynicki–Birula decompositions in the sense of Definition 3.

Białynicki–Birula theory for holomorphic $\mathbb{C}^*$ actions on complex analytic spaces

We have the following generalization of Definition 3.

Definition 5 (Białynicki–Birula decompositions for complex analytic spaces)

Let $(X, \mathscr{O}_X)$ be a complex analytic space and $\mathbb{C}^* \times X \to X$ be a holomorphic action such that the subset $X^0 \subset X$ of fixed points of the $\mathbb{C}^*$ action is non-empty with at most countably many connected components, X^0_α for $\alpha \in \mathscr{A}$, that are locally closed complex analytic subspaces of X. For each $\alpha \in \mathscr{A}$, define $X^\pm_\alpha$ as in (12) and the natural maps $\pi^\pm_\alpha$ as in (13). Then X has a *(mixed, plus, or minus) Białynicki–Birula decomposition* if the following hold:

1. Each X^+_α is a locally closed, complex analytic subspace of X;
2. The map $\pi^+_\alpha : X^+_\alpha \to X^0_\alpha$ is a $\mathbb{C}^*$-equivariant epimorphism of complex analytic spaces;

and the analogous properties hold for the subsets X^-_α and for the maps $\pi^-_\alpha : X^-_\alpha \to X^0_\alpha$.

Definition 5 (Białynicki–Birula decompositions for complex analytic spaces)

Furthermore, we require that:

3. X may be expressed as a disjoint union as in (15), for some subset $A \subset \mathscr{A} \times \{+, -\}$.

If one can express X as a disjoint union as in (16), then we say that X has a *plus* or *minus decomposition*, respectively and, otherwise, if X is expressed as in (15), that it has a *mixed decomposition*.

Weber [59, Section 2, p. 539] studied the Białynicki–Birula decomposition for singular complex algebraic varieties with $\mathbb{C}^*$ actions that are $\mathbb{C}^*$-equivariantly embedded in smooth, complex algebraic varieties.

Drinfeld provides a more general framework in a 2013 preprint [18].

One can define analogues of Morse–Bott index, co-index, and nullity:

Definition 6 (Stable and unstable subspaces of a complex analytic space and Białynicki–Birula index, co-index, and nullity)

The locally closed complex analytic subspace X_α^+ (respectively, X_α^-) is called the *stable* (respectively, *unstable*) *subspace* for the fixed-point subspace X_α^0. For each point $p \in X_\alpha^0$, the Krull dimensions $\beta_X^0(p)$, $\beta_X^+(p)$, and $\beta_X^-(p)$, of the local rings $\mathscr{O}_{X_\alpha^0,p}$, $\mathscr{O}_{X_p^+,p}$, and $\mathscr{O}_{X_p^-,p}$ are called the *Białynicki–Birula nullity*, *co-index*, and *index*, respectively, of the point p in X defined by the $\mathbb{C}^*$ action, where we write $X_p^\pm = X_\alpha^\pm|_p$ for the fibers of $X_\alpha^\pm$ over a point $p \in X_\alpha^0$.

We have the following generalization of Theorem 4.

Theorem 7 (Białynicki–Birula decomposition for a complex analytic space)

Let X be a finite-dimensional complex manifold, $(Y, \mathscr{O}_Y)$ be a closed complex analytic subspace of X, and $\mathbb{C}^* \times X \to X$ be a holomorphic action on X that leaves Y invariant with at least one fixed point in Y.

1. *If X has a plus (respectively, minus or mixed) Białynicki–Birula decomposition as in Definition 3 with subsets X^0, $X^\pm$, $X^\pm_\alpha$, then Y inherits a plus (respectively, minus or mixed) Białynicki–Birula decomposition as in Definition 5 with locally closed complex analytic subspaces:*

$$Y^0 = Y \cap X^0, \quad Y^\pm = Y \cap X^\pm, \quad Y^\pm_p = Y \cap X^\pm_p, \quad \text{for all } p \in Y^0. \tag{17}$$

2. *If $p \in Y$, then there is an open neighborhood $U \subset Y$ of p such that $\dim(Y_{\mathrm{sm}} \cap U) = \dim \mathscr{O}_{Y,p}$, where $Y_{\mathrm{sm}} \subset Y$ is the subset of smooth points.*

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Theorem 7 (Białynicki–Birula decomposition for a complex analytic space)

3. *If $p \in Y^0$ then, after possibly shrinking U, the Białynicki–Birula nullity, co-index, and index of p in Y in the sense of Definition 6 are given by*

$$\beta_Y^0(p) = \dim \mathscr{O}_{Y^0,p} = \dim(Y^0)_{\mathrm{sm}} \cap U, \tag{18a}$$

$$\beta_Y^+(p) = \dim \mathscr{O}_{Y_p^+,p} = \dim(Y_p^+)_{\mathrm{sm}} \cap U, \tag{18b}$$

$$\beta_Y^-(p) = \dim \mathscr{O}_{Y_p^-,p} = \dim(Y_p^+)_{\mathrm{sm}} \cap U, \tag{18c}$$

where $(Y^0)_{\mathrm{sm}} \subset Y^0$ and $(Y_p^\pm)_{\mathrm{sm}} \subset Y_p^\pm$ denote subsets of smooth points.

4. *If $\beta_Y^0(p) > 0$ (respectively, $\beta_Y^+(p) > 0$ or $\beta_Y^-(p) > 0$), then $(Y^0)_{\mathrm{sm}} \cap U$ (respectively, $(Y_p^+)_{\mathrm{sm}} \cap U$ or $(Y_p^-)_{\mathrm{sm}} \cap U$) is non-empty.*

5. *If the induced circle action $S^1 \times X \to X$ has a Hamiltonian function $f : X \to \mathbb{R}$ and $\beta_Y^-(p) > 0$ (respectively, $\beta_Y^+(p) > 0$), then p is not a local minimum (respectively, maximum) of the restriction $f : Y_{\mathrm{sm}} \cup \{p\} \to \mathbb{R}$.*

Krull dimensions are difficult to compute, but they may be estimated.

Suppose $(U, \mathscr{O}_U)$ is a local model space for an open neighborhood of a point p in a complex analytic space $(X, \mathscr{O}_X)$, so U is the topological support of $\mathscr{O}_D/\mathscr{I}$ with a domain $D \subset \mathbb{C}^n$ and ideal $\mathscr{I} \subset \mathscr{O}_D$ with generators $f_1, \dots, f_r$ and structure sheaf $\mathscr{O}_U := (\mathscr{O}_D/\mathscr{I}) \restriction U$. One has

$$\dim \mathscr{O}_{X,p} \geq n - r,$$

where $\exp\dim_p X := n - r$ is the expected dimension of X at p. When r is equal to the *minimal number* of generators of $\mathscr{I}_p \subset \mathscr{O}_{U,p}$, then

$$\dim \mathscr{O}_{X,p} = n - r.$$

Local models for moduli spaces are Kuranishi models and expected dimensions are computable via the Hirzebruch–Riemann–Roch Theorem.

Such a lower bound for the Białynicki–Birula index is called the virtual Białynicki–Birula index (or virtual Morse–Bott index).

Main results

Theorem 8 (Feasibility of spin^u structures)

(See F. and Leness [21, Theorem 3].) Let X be a standard four-manifold with $b^-(X) \geq 2$ and Seiberg–Witten simple type. Let $\widetilde{X} = X \# \overline{\mathbb{CP}}^2$ denote the smooth blow-up of X and let $\tilde{g}$ be a smooth Riemannian metric on $\widetilde{X}$. Then there exists a complex rank two vector bundle E over $\widetilde{X}$ and spin^u structure $\tilde{\mathfrak{t}} = (\rho, W \otimes E)$ over $\widetilde{X}$ such the following hold:

1. *The moduli space $\mathscr{M}_{\tilde{\mathfrak{t}}}^{*,0}$ of irreducible, non-zero section non-Abelian monopoles is non-empty for generic Riemannian metrics.*
2. *The bundle E over $\widetilde{X}$ obeys the basic lower bound,*

$$p_1(\mathfrak{su}(E)) \geq c_1(\widetilde{X})^2 + 1 - 12\chi_h(\widetilde{X}). \tag{19}$$

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Theorem 8 (Feasibility of spin^u structures)

3. *The expected dimension of the moduli space $M_\kappa^w(\widetilde{X}, \tilde{g})$ of anti-self-dual connections obeys the following inequality:*

$$\frac{1}{2} \exp \dim M_\kappa^w(\widetilde{X}, \tilde{g}) \leq -c_1(X)^2 + 9\chi_h(X). \tag{20}$$

4. *For all spin^c structures $\tilde{\mathfrak{s}}$ for which $M_{\tilde{\mathfrak{s}}}$ continuously embeds in $\mathscr{M}_{\tilde{\mathfrak{t}}}$, the formal Morse–Bott index is positive:*

$$\lambda^-(\tilde{\mathfrak{t}}, \tilde{\mathfrak{s}}) := -2\chi_h(\widetilde{X}) - \left(c_1(\tilde{\mathfrak{s}}) - c_1(\tilde{\mathfrak{t}})\right) \cdot c_1(\widetilde{X}) - \left(c_1(\tilde{\mathfrak{s}}) - c_1(\tilde{\mathfrak{t}})\right)^2 > 0. \tag{21}$$

The expression (21) has the following motivation.

Suppose that $[A, \Phi]$ is a reducible, type 1 non-Abelian monopole with $\Phi \not\equiv 0$, thus a fixed point of the S^1 action on $\mathscr{M}_{\mathfrak{t}}$ and a critical point of the restriction of Hitchin's Hamiltonian function f on the configuration space $\mathscr{C}_{\mathfrak{t}}^0$ of all non-zero section, unitary pairs in the sense that

$$H^1_{A,\Phi} = T_{A,\Phi}\mathscr{M}_{\mathfrak{t}} \subseteq \operatorname{Ker} df[A,\Phi].$$

The point $[\bar\partial_A, \varphi] \in \mathfrak{M}^\mu(E)$ defined by the Hitchin–Kobayashi correspondence, where $\bar\partial_A$ is a holomorphic structure on E and $\Phi = (\varphi, 0) \in \Omega^0(E) \oplus \Omega^{0,2}(E)$, is a fixed point of the $\mathbb{C}^*$ action on $\mathfrak{M}^\mu(E)$.

One has the following weight splittings of Zariski tangent spaces and obstruction spaces defined by the S^1 action:

$$H^1_{A,\Phi} = H^{0,1}_{A,\Phi} \oplus H^{+,1}_{A,\Phi} \oplus H^{-,1}_{A,\Phi}, \quad \text{and} \quad H^2_{A,\Phi} = H^{0,2}_{A,\Phi} \oplus H^{+,2}_{A,\Phi} \oplus H^{-,2}_{A,\Phi},$$
$$H^1_{\bar\partial_A,\varphi} = H^{0,1}_{\bar\partial_A,\varphi} \oplus H^{+,1}_{\bar\partial_A,\varphi} \oplus H^{-,1}_{\bar\partial_A,\varphi}, \quad \text{and} \quad H^2_{\bar\partial_A,\varphi} = H^{0,2}_{\bar\partial_A,\varphi} \oplus H^{+,2}_{\bar\partial_A,\varphi} \oplus H^{-,2}_{\bar\partial_A,\varphi}.$$

One has $H^0_{A,\Phi} = (0)$ and $H^0_{\bar\partial_A,\varphi} = (0)$ since $\varphi \not\equiv 0$ and one can also show that $H^3_{\bar\partial_A,\varphi} = (0)$.

Euler characteristics of the negative-weight complexes, $H^{-,\bullet}_{A,\Phi}$ and $H^{\bullet,-}_{\bar\partial_A,\varphi}$, are equal and computable by the Hirzebruch–Riemann–Roch Theorem:

$$\mathrm{Euler}\left(\bar\partial^{-,\bullet}_{A,\varphi}\right) := \sum_{k=0}^{3}(-1)^k H^{-,k}_{\bar\partial_A,\varphi} = \sum_{k=0}^{3}(-1)^k H^{-,k}_{A,\Phi} =: \mathrm{Euler}\left(d^{-,\bullet}_{A,\Phi}\right).$$

Hence, the virtual Morse–Bott index for f at $[A,\Phi] \in \mathscr{M}_{\mathfrak{t}}$ or equivalently, the virtual Białynicki–Birula index for the fixed point $[\bar\partial_A,\varphi] \in \mathfrak{M}^{\mu}(E)$ of the $\mathbb{C}^*$ action ρ are given by

$$\begin{aligned}\lambda^-_{[A,\Phi]}(f) &:= H^{-,1}_{A,\Phi} - H^{-,2}_{A,\Phi} = -\,\mathrm{Euler}\left(d^{-,\bullet}_{A,\Phi}\right)\\ &= -\,\mathrm{Euler}\left(\bar\partial^{-,\bullet}_{A,\varphi}\right) = H^{-,1}_{\bar\partial_A,\varphi} - H^{-,2}_{\bar\partial_A,\varphi} =: \lambda^-_{\bar\partial_A,\varphi}(\rho). \quad (22)\end{aligned}$$

The forthcoming Theorem 9 and Corollary 10 show that the expression $\lambda^-(\tilde{\mathfrak{t}},\tilde{\mathfrak{s}})$ in (21) is *equal* to the virtual Morse–Bott index $\lambda^-_{[A,\Phi]}(f)$.

Theorem 9 (Virtual Morse–Bott index of Hitchin's Hamiltonian function at a reducible non-Abelian monopole)

(See F. and Leness [21, Theorem 5].) Let $(\rho_{\mathrm{can}}, W_{\mathrm{can}})$ *be the canonical* spin^c *structure over a closed, complex Kähler surface* X*, and* E *be a complex rank two Hermitian vector bundle over* X *that admits a splitting* $E = L_1 \oplus L_2$ *as a direct sum of Hermitian line bundles, and* $\mathfrak{t} = (\rho, W_{\mathrm{can}} \otimes E)$ *be the corresponding* spin^u *structure. Assume that* (A, Φ) *is a type* 1 *non-Abelian monopole on* $\mathfrak{t}$ *that is reducible with respect to the splitting* $E = L_1 \oplus L_2$ *as a direct sum of Hermitian line bundles, with* $\Phi = (\Phi_1, 0)$ *and* $\Phi_1 \in \Omega^0(W_{\mathrm{can}}^+ \otimes L_1)$ *non-zero. Then the virtual Morse–Bott index* (22) *of Hitchin's Hamiltonian function* f *in* (10) *at the point* $[A, \Phi] \in \mathscr{M}_{\mathfrak{t}}$ *is given by minus twice the Euler characteristic of the negative-weight elliptic complex for the holomorphic pair* $(\bar\partial_A, \varphi)$*, where* $\Phi_1 = (\varphi, 0) \in \Omega^0(L_1) \oplus \Omega^{0,2}(L_1)$*, and equals*

$$\lambda^-_{[A,\Phi]}(f) = -2\chi_h(X) - (c_1(L_1) - c_1(L_2)) \cdot c_1(X) - (c_1(L_1) - c_1(L_2))^2 . \quad (23)$$

Corollary 10 (Virtual Morse–Bott index of Hitchin's Hamiltonian function at a point represented by a Seiberg–Witten monopole)

(See F. and Leness [21, Corollary 6].) Let X be a closed, complex Kähler surface, $\mathfrak{t}$ be a spin^u structure over X, and $\mathfrak{s}$ be a spin^c structure over X. If $[A, \Phi] \in \mathscr{M}_{\mathfrak{t}}$ is a point represented by a reducible, non-Abelian type 1 monopole in the image of the embedding of the moduli space $M_{\mathfrak{s}}$ of Seiberg–Witten monopoles on $\mathfrak{s}$ into the moduli space $\mathscr{M}_{\mathfrak{t}}$ of non-Abelian monopoles on $\mathfrak{t}$, then the virtual Morse–Bott index (22) of Hitchin's Hamiltonian function f in (10) on the moduli space $\mathscr{M}_{\mathfrak{t}}$ at $[A, \Phi]$ is given by

$$\lambda^{-}_{[A,\Phi]}(f) = -2\chi_h(X) - (c_1(\mathfrak{s}) - c_1(\mathfrak{t})) \cdot c_1(X) - (c_1(\mathfrak{s}) - c_1(\mathfrak{t}))^2, \qquad (24)$$

where $c_1(\mathfrak{s}) := c_1(W^+) \in H^2(X; \mathbb{Z})$ and $c_1(\mathfrak{t}) := c_1(E) + c_1(W^+) \in H^2(X; \mathbb{Z})$.

Corollary 11 (Positivity of virtual Morse–Bott index of Hitchin's Hamiltonian at point represented by Seiberg–Witten monopole for a feasible spin^u structure)

(See F. and Leness [21, Corollary 7].) Let X be a closed, complex Kähler surface with $b_1(X) = 0$, $b^-(X) \geq 2$, and $b^+(X) \geq 3$. If $\widetilde{X} = X \# \overline{\mathbb{CP}}^2$ is the blow-up of X and $\tilde{\mathfrak{t}}$ is the spin^u structure on $\tilde{X}$ constructed in Theorem 8, then for all non-empty Seiberg–Witten moduli subspaces $M_{\tilde{\mathfrak{s}}}$ that are continuously embedded in $\mathscr{M}_{\tilde{\mathfrak{t}}}$ as type 1 reducible, non-Abelian monopoles, the virtual Morse–Bott index (22) of Hitchin's Hamiltonian f in (10) on $\mathscr{M}_{\tilde{\mathfrak{t}}}$ is positive at all points in $M_{\tilde{\mathfrak{s}}}$.

Next steps

We have been exploring two main directions:

Uhlenbeck bubbling

Extend our calculation of virtual Morse–Bott indices for points $[A, \Phi] \in \mathscr{M}_\mathfrak{t}$ represented by Seiberg–Witten monopoles to points $[A, \Phi] \in \bar{\mathscr{M}}_\mathfrak{t}$ that lie in $\mathrm{Sym}^\ell(X) \times \mathscr{M}_{\mathfrak{t}(\ell)}$, that is, allow for bubbling.

There are two ways to address this:

1. Use gluing to construct local models for Uhlenbeck boundary points.
2. Replace $\bar{\mathscr{M}}_\mathfrak{t}$ by the Gieseker compactification $\mathfrak{M}^{ss}(E)$.

Both approaches should lead to gauge-theoretic proofs of the Bogomolov–Miyaoka–Yau inequality for complex surfaces of general type.

The gluing method has potential to extend the proof to smooth four-manifolds of Seiberg–Witten simple type.

We expect that the virtual Morse–Bott index of a Seiberg–Witten fixed point decreases as an instanton bubble of multiplicity ℓ forms, but a version of the Bogomolov–Gieseker inqualty due to Bradlow [11, Theorem 4.1, p. 208] appears to prevent ℓ from becoming so large that the virtual Morse–Bott index of a Seiberg–Witten fixed point is non-positive.

Non-integrable almost complex structures and non-Kähler metrics

Extend our calculation of virtual Morse–Bott indices when X is a Kähler surface by allowing X to be an almost Hermitian, smooth four-manifold.

For this purpose, a conjecture due to Tobias Shin [23] appears useful:

> *If (X, g, J) is an almost Hermitian manifold, then there is a sequence $\{J_i\}_{i\in\mathbb{N}}$ of almost complex structures J_i on X such that the C_0-norms of the Nijenhuis tensors N_{J_i} become arbitrarily small as $i \to \infty$.*

Shin's conjecture holds for examples of four-manifolds that are almost complex but not complex.

If Shin's conjecture is true, methods of *holomorphic approximation* (in the spirit of Auroux, Donaldson, and Taubes) in combination with *gluing* should allow us to extend our calculations to the general case of Conjecture 1, where X is a standard four-manifold of Seiberg–Witten simple type.

Thank you for your attention!

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