@MSRI, Nov. 9 2022.

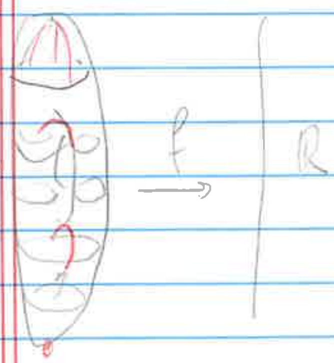
Morse th. for geo. (on S^n)

Old subject, much remains.

Ref: Donaldson, "Atiyah's works", last section.

Morse th. $\equiv$ Morse-Smale(-Witten) th.

$\mathrm{Hess}_x f \neq 0$ for $x \in \mathrm{Crit} f$.



$\mathrm{Uns}_x f \pitchfork \mathrm{St}_y f$ $\forall x, y \in \mathrm{Crit} f$.

$\lambda_x = \#$ neg. evals of $\mathrm{Hess}_x f$ for $x \in \mathrm{Crit} f$

"morse index"

Main thm: X is homot. equiv. to CW cpx w/ one cell in dim λ_x for each $x \in \mathrm{Crit} f$.

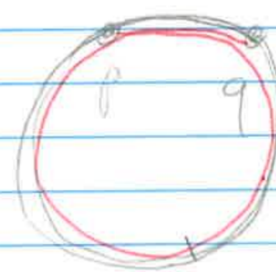
Moral: Topol. of X $\longleftrightarrow$ Crit. pts. of f.

Used to think of a more fund./profound idea in geom. anal.

Morse applied to geodesics!

$\mathbb{R}^3 \times S^1 \simeq \mathbb{R}^2 \times \mathbb{C}^{\times} \simeq S^4 \setminus S^2$

$$H_*(\Omega_{p,q} S^n) \cong \underset{0}{\mathbb{Z}}, 0, \ldots, 0, \underset{n-1}{\mathbb{Z}}, 0, \ldots, 0, \underset{2(n-1)}{\mathbb{Z}}, \ldots$$



p, q not conjugate.

$\Rightarrow$ for any metric on S^n, p, q not conjugate, $\exists$ ∞'ly many geodesics from p to q.

Pf uses f.-d. approx.

how to work directly on ∞-d?

$$X \rightsquigarrow \mathcal{B} \text{ — configuration sp.}$$

$$f \rightsquigarrow E \text{ — energy fcl.}$$

Examples:

- $\mathcal{B} = \mathrm{Maps}(\Sigma, X)$, $E = \frac{1}{2}\int |\nabla u|^2$ → harmonic maps.
- $\mathcal{B}$ = integral currents, E = area — min'l surfaces
- $\mathcal{B} = \mathcal{A}/\mathcal{G}$, $E = YM(A) = \frac{1}{2}\int |F_A|^2$ — YM conn's.

Difficulties

1. Manifold str. on $\mathcal{B}$?
2. Morse-ness of E?
3. attaching maps / diff'ble?
4. Properness of E? E can never be proper! on ∞-d. space
5. Exist & Conv. of gradient flow?

Atiyah-Bott '82: YM on R-surf ($n=2$) $X=\Sigma^g$

$D_A \underbrace{*F_A}_{0\text{-form}} = 0 \Rightarrow \nabla_A * F_A = 0$

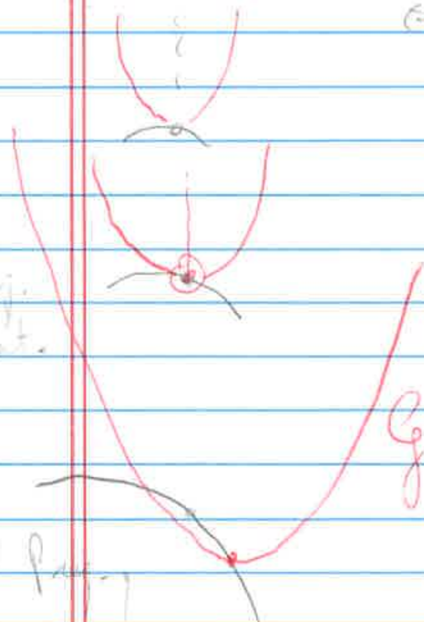
$\Rightarrow F_A = \bigoplus_i \lambda_i \cdot Id \otimes \omega$.

Obs: A minimizes YM

$\Leftrightarrow \lambda_i = \lambda_j \; \forall i,j$

$F_A = \underbrace{\mu}_{\frac{\deg E}{\mathrm{rk} E}} Id \otimes \omega$, proj. flat. ($Vol(\Sigma)=1$)

⊕ proj. flat.



$\mathcal{G}^c$-orbits.

$\{\text{Proj. flat}\}/\mathcal{G} \underset{N-S}{\overset{\text{hol.}}{\cong}} \{\text{stable bundles}\}/\mathcal{G}^c$

Topology of $\mathcal{B}$ is known a.p.

$\Rightarrow$ A-B calculate Poinc. poly. of mod. sp. of stable bundles by induction on rank.

Base case = Hodge Thm for $\bar\partial$-operator.

(Atiyah-Bott's)

<u>Solutions to difficulties</u>

1. $\mathcal{B} \rightsquigarrow \mathcal{B}'$ = framed mod. sp., Sobolev topol.

$H^*(\mathcal{B}) \rightsquigarrow H^*_G(\mathcal{B}') := H^*(EG \times_G \mathcal{B}')$.

2. Morse $\rightsquigarrow$ Morse-Bott

3. All diffle varieties!

Perfection theorem from self-completing principle.

(4-5. A-B use A.C. method.)

Can only work in ex-d!

4. Properness $\rightsquigarrow$ Palais-Smale <u>Cond. C</u>:

$x_i \in \mathrm{Crit}\, E$ w/ $\nabla E_{x_i} \to 0$, $\sup E(x_i) < \infty$

$\Rightarrow \exists\, x_\infty \in \mathrm{Crit}\, E$ w/ $\nabla E_{x_\infty} = 0$, $x_i \to x_\infty$

$\Rightarrow$ compactness of critical sets, (Discrete version of energy spectrum w/ real-analyticity of E)

<u>Uhl's strong compactness thm.</u>

5. L.t.e. & conv. of $\partial_t x = -\nabla E_x$ "Y-M flow"

6. Developments of J. Råde ($n = 2, 3$)

<u>$n = 4$:</u> Expect 2-5 to <u>fail</u> in general.

Still enormous amount can be done.

<u>Taubes</u>.

monopoles = S^1-equiv. inst. on $\mathbb{R}^3 \times S^1$

YM ⤳ YMH (crit., 2nd-order)

dual ⤳ Bogo. E. (min., 1st-order)

Strong optimism holds unless energy escapes to ∞.

Taubes constr. crit. of YMH > Bogom.

by min-max on $[\gamma] \neq 0 \in \pi_1(\mathcal{B})$.

Need $E(A_- \#_{\gamma(\theta)} A_+) < 2 \quad \forall\, \theta \in S^1$

to avoid bubbling 'modes'.

$X = S^4, \quad G = SU(2), \quad k = c_2.$

$\mathcal{M}_k^1 = \{F^+ = 0\}/\mathcal{G}^1 \subset \mathcal{B}_k^1$

Questions: 1. What is the topology of $\mathcal{M}_k$?

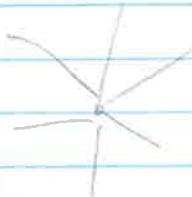
2. What are the higher crit. pts of YM?

($\exists$ any?)

Point: should be related.

Analogy: Hopf's $S^3 \to S^2$ ...

$\mathcal{B}'_k \simeq \mathrm{Map}_k(S^3, S^3) \simeq \Omega^3_k S^3$



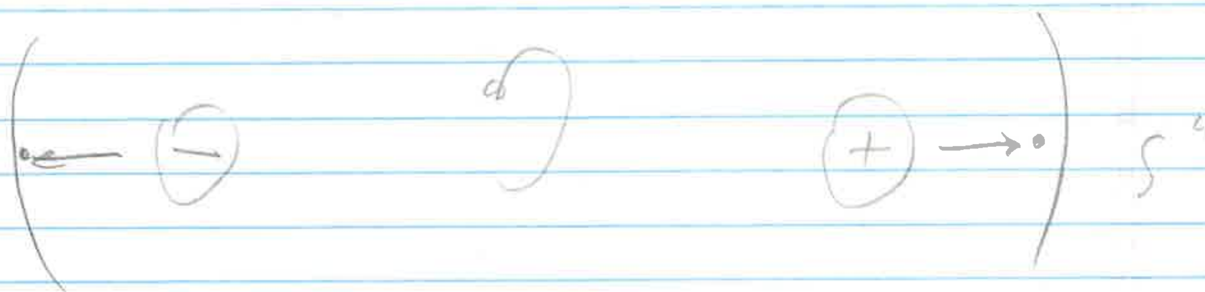
Fact: $\pi_1(\mathcal{B}_k) \cong \mathbb{Z}_2$ for k even.

But Taubes proved: For non-min. YM,

Morse index $\geq 2(|k|+1)$!

Min-max over $[\gamma]$:

must bubble. (Can't make $E<2$.)



See ideal Y-M conn. of "index" $= 1$!

Taubes tried to include — "A framework for Morse Th."

Partial answers:

1. $\exists$ homot. equiv.

$$\begin{tikzcd} \mathcal{B}'_k \arrow[r, "\sim"] & \mathcal{B}'_{k+1} \\ \mathcal{M}'_k \arrow[u] \arrow[r, "\text{Taub.}"] & \mathcal{M}'_{k+1} \arrow[u] \arrow[r] & \cdots \end{tikzcd}$$

Atiyah-Jones Conj: $\pi_i(\mathcal{M}'_k) \xrightarrow{\simeq} \pi_i(\mathcal{M}'_{k+1})$ for $i < q(k) \to \infty$ as $k \to \infty$.

Proved on S^4 by B-H-M-M. Q. General 4-mfds?

2. Sibner-Sibner-Uhlenbeck '89: constr. non-min'l YM w/ $k=0$, $2<E<4$ using thm 2

monopoles on $\mathbb{H}^3$. $\qquad \mathbb{H}^3\times S^1 \cong \mathbb{R}^2\times\mathbb{C}^* \supset S^1$

Q. Is it the lowest? $\qquad = S^4\setminus S^2$.

$\qquad\qquad\qquad\qquad\qquad\qquad \cap$

c.f. Willmore conj. $\qquad\qquad\qquad S^4$