**STUDY GROUP ON SHARIFI'S CONJECTURE**

*SLMath, February-March 2023*

# 1 Summary

The aim of this study group is giving a fairly general introduction to Sharifi's conjecture, firstly formulated in [S11], and with a special emphasis on the developments done by Fukaya and Kato [FK12]. Beyond the aforementioned sources, we will use mainly the survey paper written by all three authors on the topic [FKS14], and also Sharifi's expository notes for the Arizona Winter School [S18].

The conjecture draws a tantalizing connection between modular symbols and the arithmetic of units in cyclotomic fields. The starting point is the explicit map

$$[u : v] \mapsto \{1 - \zeta_N^u, 1 - \zeta_N^v\}, \tag{1}$$

where $N$ is a positive integer; $\zeta_N$ is a primitive $N$-th root of unity; $u, v \in \mathbb{Z}/N\mathbb{Z}$ are nonzero numbers with $(u, v) = 1$; $[u : v]$ is a Manin symbol in the relative homology group $H_1(X_1(N), \{\text{cusps}\}, \mathbb{Z})$; and $\{1-\zeta_N^u, 1-\zeta_N^u\}$ is a Steinberg symbol in the algebraic $K$-group $K_2(\mathbb{Z}[\zeta_N, \frac{1}{N}])$. Since the above map connects the geometric theory of $\mathrm{GL}_2$ with the arithmetic theory of $\mathrm{GL}_1$, there are different kind of expected generalizations linking the geometric theory of $\mathrm{GL}_d$ modulo the Eisenstein ideal and the arithmetic theory of $\mathrm{GL}_{d-1}$ over global fields. This is extensively developed in [FKS14, §3], but we will not deal with this in the study group. The appearance of the Eisenstein ideal is one of the interesting points: the homology group has the action of a Hecke algebra, and the map we described in (1) factors through the quotient of the homology by this ideal.

Fix now a prime $p \geq 5$ and consider the quotient

$$P_r = H_1(X_1(p^r), \mathbb{Z}_p)^+ / I_r H_1(X_1(p^r), \mathbb{Z}_p)^+,$$

where the signs $\pm$ stand for the corresponding eigenspace under the action of complex conjugation and $I_r$ for the corresponding Eisenstein ideal. Define the inverse limit $P = \lim_{\leftarrow r} P_r$. Let $A_r$ be the $p$-part of the class group of $\mathbb{Q}(\zeta_{p^r})$, which is isomorphic to $H^2_{\text{et}}(\mathbb{Z}[\zeta_{p^r}, \frac{1}{p}], \mathbb{Z}_p(2))^+$. Letting $X = \lim_{\leftarrow} A_r$, the map in (1) induces an application

$$\varpi : P \longrightarrow X^-(1). \tag{2}$$

The aim of the first talk will be providing a proper description of it.

Based on the works of Mazur–Wiles and Rubin, one can also consider a map

$$\Upsilon : X^-(1) \longrightarrow P, \tag{3}$$

whose definition is less straightforward and involves the Galois action on the projective limit of the reduction of étale homology groups $H_1(X_1(p^r)_{\bar{\mathbb{Q}}}, \mathbb{Z}_p)$ modulo the Eisenstein ideal. Then, Sharifi conjectures that $\varpi$ and $\Upsilon$ are isomorphisms, and they are inverse to each other. The definition of $\Upsilon$ and the formulation of Sharifi's conjecture will be the topic of the second talk.

In the last talk, we will explore the approach of Fukaya and Kato to the conjecture. Following their conventions, let $\xi'$ be the derivative of the $p$-adic Riemann zeta function. Then, after passing to suitable isotypic components, they prove that

$$\xi' \Upsilon \circ \varpi = \xi' \tag{4}$$

modulo $p$-torsion in $P$. In particular, if $\xi'$ has no multiple roots, the theorem implies the conjecture up to $p$-torsion in $P$. The proof, whose main ideas are developed in [FK12, §9], is quite technical, and requires a good Galois cohomological understanding of $X^-(1)$, the *evaluation-at-infinity* map used to define the application $\varpi$ and some deep connections between the derivative of the $p$-adic zeta function and the map $\Upsilon$.

## 2 Organization

The study group will consist on 3 talks. The following table summarizes the schedule and the list of speakers.

| Session | Day | Speaker |
|---|---|---|
| **1** | February 15th | Oscar |
| **2** | February 22nd | Gyujin |
| **3** | March 1st | Zhiyu |

## 3 Contents

- **Session 1. General overview and construction of the map $\varpi$.**
  - Overview of the topic and motivation.
  - Preliminaries about Beilinson elements, both in the $K_2$ of modular curves and in Galois cohomology.
  - Definition of the map $\varpi$ (*evaluation-at-infinity* maps).

  *Main references.* Fukaya–Kato–Sharifi: Sections 1, 2.1–2.3; Fukaya–Kato: Section 5 (and also some parts of 1–3); Sharifi's original paper; Sharifi's notes for the Arizona Winter School: Chapter 5.

- **Session 2. Construction of the map $\Upsilon$ and formulation of Sharifi's conjecture.**
  - Galois representations and the structure of $\varprojlim H^1(X_1(p^r)_{\bar{\mathbb{Q}}}, \mathbb{Z}_p)$.
  - Definition of the map $\Upsilon$.
  - Formulation of Sharifi's conjecture.

  *Main references.* Fukaya–Kato–Sharifi: Sections 2.4–2.6; Fukaya–Kato: Sections 6–7.

- **Session 3. The results of Fukaya–Kato.**
  - Recap on $p$-adic $L$-functions in two variables.
  - Study of Galois cohomology.
  - The results of Fukaya–Kato.

  *Main references.* Fukaya–Kato–Sharifi: Section 2.7; Fukaya–Kato: Sections 8, 9, 10.

## References

[FK12] T. Fukaya and K. Kato. *On conjectures of Sharifi*, preprint.

[FKS14] T. Fukaya, K. Kato and R. Sharifi. *Modular symbols in Iwasawa theory*, in Iwasawa Theory 2012, *Contributions in Mathematical and Computational Sciences* (2014).

[S11] R. Sharifi. *A reciprocity map and the two-variable p-adic L-function*, Annals of Mathematics (2011).

[S18] R. Sharifi. *Modular curves and cyclotomic fields*, notes for the Arizona Winter School 2018 notes, available online.