(Oscar Rivero, 15-feb-2023)

SHARIFI'S CONJECTURE I: FROM MODULAR SYMBOLS TO CUP PRODUCTS

§1. Motivation

We begin with the correspondence

$$[u:v] \longmapsto \{1-\zeta_N^u, 1-\zeta_N^v\},$$

where $N \geq 1$; ζ_N is an N-th root of unity; $u, v \in \mathbb{Z}/N\mathbb{Z}$; $[u:v]$ is the Manin symbol in the real homology group $H_1(X_1(N), \{\text{cusps}\}, \mathbb{Z})$ (to be defined); and $\{1-\zeta_N^u, 1-\zeta_N^v\} \in K_2(\mathbb{Z}[\zeta_N, \frac{1}{N}])$ is the Steinberg symbol.

[Philosophy] geom. theory $GL_2/_{\text{Eis. ideal}} \longrightarrow$ arith. theory GL_1

$\rightsquigarrow$ modulo the Eisenstein ideal we will get isomorphism (in a precise way).

[Aim] geom. theory $GL_d/_{\text{Eis. ideal}} \longleftrightarrow$ arith. theory GL_{d-1}

(over general global fields).

[in this section we are oversimplifying: we'll need to invert 2 + take the (+1)-eigencomponent under complex conjugation].

Big picture: work w/ $p \geq 3$ prime and level p^r, $r \geq 1$.

$$P_r = H_1(X_1(p^r), \mathbb{Z}_p)^+ / \underbrace{I_r}_{\text{Eis. ideal level } p^r} H_1(X_1(p^r), \mathbb{Z}_p)^+$$

Want: map $\varpi_r : P_r \longrightarrow H^2_{et}(\mathbb{Z}[\zeta_{p^r}, \frac{1}{p}], \mathbb{Z}_p(2))^+$

$$[u,v] \longmapsto \{1-\zeta_{p^r}^u, 1-\zeta_{p^r}^v\}$$

$$H^2_{et}(\mathbb{Z}[\zeta_{p^r}, \tfrac{1}{p}], \mathbb{Z}_p(2)) \otimes \mathbb{Z}/p^r\mathbb{Z} \cong A_r^-/p^r A_r^- (1)$$

(A_r p-part of class group of $\mathbb{Q}(\zeta_{p^r})$).

$P = \varprojlim P_r$, $X = \varprojlim A_r$

$$\varpi : \underbrace{P}_{\text{tower of mod. curves } X_1(p^r)} \longrightarrow \underbrace{X^-(1)}_{\text{arithmetic of cycl. fields}}$$

[Conj.] $\varpi : P \longrightarrow X^-(1)$ is an isomorphism

Υ next talk: defined via Galois action on the proj. limit of étale homology groups (modulo Eis. ideal).

[Iwasawa main conjecture] char$(X^-) = (\zeta_p)$

Since char$(P) = (\zeta_p(1))$, proving that char$(P) =$ char$(X^-(1))$ is enough to have the IMC. Hence, the conjecture can be seen as a strengthening of the IMC.

§2. From modular symbols to cup products

- $p \geq 3$ prime, $N \geq 1$ w/ $p \nmid N$. $r \geq 1$.
- Fix $\overline{\mathbb{Q}}$ + embedding $\overline{\mathbb{Q}} \hookrightarrow \overline{\mathbb{Q}_p}$.
- $\Gamma_1(Np^r)$, $Y_1(Np^r) = \Gamma_1(Np^r) \backslash \mathbb{H}$.
- Write C_r for the cusps. $X_1(Np^r) = Y_1(Np^r) \cup C_r$.

$$S_r = H_1(X_1(Np^r), \mathbb{Z})_+ \; ; \qquad M_r = H_1(X_1(Np^r), C_r, \mathbb{Z})_+$$

(+ means (+1)-eigenspace under complex conjugation)

Manin symbols $[u:v] \in M_r$.
$u, v \in \mathbb{Z}/_{Np^r\mathbb{Z}}$ w/ $(u,v) = 1$. Fix $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{Z})$
w/ $u \equiv c$, $v \equiv d$ (Np^r).

$$[u:v]_r = \left\{ \frac{d}{bNp^r} \longrightarrow \frac{c}{aNp^r} \right\}$$

w/ $\{\alpha \to \beta\}$ class in M_r of the hyperbolic geodesic on $\mathbb{H}^*$.

(M_r is the abelian group w/ generators $[u,v]_r$ and relations $[u:v]_r = [-u:v]_r = -[v:u]_r$,
$[u:v]_r = [u:u+v]_r + [u+v:v]_r$.)

$S_r \subset M_r^\circ \subset M_r$, where $M_r^\circ = H_1(X_1(Np^r), C_r^\circ, \mathbb{Z})_+$
being C_r° the cusps that do not lie over the 0-cusp.

ζ_{Np^r}, $E_r = \mathbb{Q}(\zeta_{Np^r})$, $F_r = \mathbb{Q}(\zeta_{Np^r})^+$, $\mathcal{O}_r = \mathbb{Z}[\zeta_{Np^r}]^+$.

Aside / definition: R comm. ring. The second Milnor K-group is $K_2(R) := (R^\times \otimes_{\mathbb{Z}} R^\times) / I_2$, with
$I_2 = \langle a_1 \otimes a_2 \in R^\times \otimes_{\mathbb{Z}} R^\times \mid a_1 + a_2 \in \{0,1\} \rangle$.
The Steinberg symbol is the canonical map

$$\{ , \} : R^\times \times R^\times \longrightarrow K_2(R)$$

Easy to check that $\{a,b\} = -\{b,a\}$ and for $x, y \in \mathbb{Z}/_{p^n\mathbb{Z}}$
w/ $x \neq 0$ we have $\{1 - \zeta_M^x, \zeta_M^y\} = 0$.

Prop: $\exists$ a well-defined homomorphism

$$\varpi_r : M_r^o \otimes \mathbb{Z}[\tfrac{1}{2}] \longrightarrow K_2(\mathcal{O}_r[\tfrac{1}{Np}]) \otimes \mathbb{Z}[\tfrac{1}{2}])$$
$$[u:v] \longmapsto \{1-\zeta_{Np^r}^u, 1-\zeta_{Np^r}^v\}$$

<u>Aside.</u> $\underbrace{\kappa : R^\times \longrightarrow H^1_{\text{ét}}(R, \mathbb{Z}/p^n\mathbb{Z}(1))}_{\text{(Kummer map)}}$ (p invertible on R)

(Chern class map) $K_2(R) \longrightarrow H^2_{\text{ét}}(R, \mathbb{Z}/p^n\mathbb{Z}(2))$
$$\{a,b\} \longmapsto \kappa(a) \cup \kappa(b)$$

$R = \mathbb{Z}[\zeta_{Np^r}, \tfrac{1}{Np}]$, $\mathcal{O}_r[\tfrac{1}{Np}]$.

$$\begin{array}{ccc} K_2(\mathbb{Z}[\zeta_{Np^r}, \tfrac{1}{Np}])_+ \otimes \mathbb{Z}_p & \xrightarrow{\sim} & H^2_{\text{ét}}(\mathbb{Z}[\zeta_{Np^r}, \tfrac{1}{Np}], \mathbb{Z}_p(2))_+ \\ N \downarrow \wr & & \text{cor} \downarrow \wr \\ K_2(\mathcal{O}_r[\tfrac{1}{Np}]) \otimes \mathbb{Z}_p & \xrightarrow{\sim} & H^2_{\text{ét}}(\mathcal{O}_r[\tfrac{1}{Np}], \mathbb{Z}_p(2)) \end{array}$$

$$\varpi_r : M_r^o \otimes \mathbb{Z}_p \longrightarrow H^2_{\text{ét}}((\mathcal{O}_r[\tfrac{1}{Np}], \mathbb{Z}_p(2))$$
$$[u:v]_r \longmapsto \{1-\zeta_{Np^r}^u, 1-\zeta_{Np^r}^v\}$$

cup product + corestriction

§3. The map ϖ

<u>Hecke algebras</u> : $\tilde{\mathbb{T}}_r \subset \mathrm{End}_{\mathbb{Z}_p}(M_r \otimes \mathbb{Z}_p)$.
$$\langle T(n) \mid n \geq 1 \rangle$$

$\mathbb{T}_r$ cuspidal Hecke algebra. They contain $\langle d \rangle$, $(d, Np) = 1$.
$\tilde{I}_r \subset \tilde{\mathbb{T}}_r$ Eisenstein ideal generated by $T(\ell) - 1 - \ell\langle \ell \rangle$
I_r image in $\mathbb{T}_r$ under $\tilde{\mathbb{T}}_r \twoheadrightarrow \mathbb{T}_r$.

[Conj. Sharifi / Thm. FK] ϖ_r factors through the Eisenstein ideal

$$(M_r^o / I_r M_r^o) \otimes \mathbb{Z}_p \longrightarrow H^2_{\text{ét}}(\mathcal{O}_r[\tfrac{1}{Np}], \mathbb{Z}_p(2))$$

$\hookrightarrow$ we'll come back later on.

Pass to the limit? $G_r = (\mathbb{Z}/Np^r\mathbb{Z})^\times/\{\pm 1\}$, $\Lambda_r = \mathbb{Z}_p[G_r]$.
$G_r \xrightarrow{\sim} \mathrm{Gal}(F_r/\mathbb{Q})$; $\iota_r : \Lambda_r \hookrightarrow \mathbb{T}_r$
$a \mapsto \langle a \rangle^{-1}$

Action compatibility $\varpi_r(\langle a\rangle^{-1} x) = \sigma_a \varpi_r(x)$.

RK.: ϖ_r is Eisenstein means $\varpi_r(T(\ell)x) = (1+\ell\sigma_\ell^{-1})\varpi_r(x)$ for $\ell \nmid Np$ and $\varpi_r(T(\ell)x) = \varpi_r(x)$ for $\ell \mid Np$.

Def.: $G = \varprojlim G_r$, $\Lambda = \varprojlim \Lambda_r$, $\mathbb{T}$, $\tilde{\mathbb{T}}$.
$K = \bigcup_{r\geq 1} F_r$, $E = \bigcup_{r\geq 1} E_r$, $\Lambda \xrightarrow{\sim} \mathbb{Z}_p[[\mathrm{Gal}(K/\mathbb{Q})]]$.

$S = \varprojlim(S_r \otimes \mathbb{Z}_p)$, $M^\circ = \varprojlim(M_r^\circ \otimes \mathbb{Z}_p)$.
$\tilde{I} \subset \tilde{\mathbb{T}}$, $I \subset \mathbb{T}$ Eis. ideals.

Prop.: $\exists$ a map $\varpi : M^\circ \longrightarrow \varprojlim_r H^2_{\acute{e}t}(\mathcal{O}_r[\frac{1}{Np}], \mathbb{Z}_p(2))$ factoring through $M^\circ/\tilde{I}M^\circ$.
Further ϖ is a homomorphism of Λ-modules.

Aside. $A_r = Cl(E_r)[p]$, $X = \varprojlim_r A_r \longrightarrow \varprojlim_r H^2_{\acute{e}t}(\mathcal{O}_r[\frac{1}{Np}], \mathbb{Z}_p(1))$
The latter induced by Kummer
$$Pic(R) = H^1_{\acute{e}t}(R, \mathbb{G}_m) \longrightarrow H^2_{\acute{e}t}(R, \mathbb{Z}/p^n\mathbb{Z}(1)).$$

neither inj. nor surjective

$X = \mathrm{Gal}(L/\mathbb{Q})$, w/ L max. unr. abelian pro-p extension.
$X = X^- \oplus X^+$; $Y = X^-(1) \circlearrowleft \Lambda$

Then, have a Λ-module homomorphism
$$Y \longrightarrow \varprojlim_r H^2_{\acute{e}t}(\mathcal{O}_r[\tfrac{1}{Np}], \mathbb{Z}_p(2))$$
θ even char. of $(\mathbb{Z}/Np\mathbb{Z})^\times$, $\Lambda_\theta = \Lambda \otimes_{\mathbb{Z}_p[(\mathbb{Z}/Np\mathbb{Z})^\times]} \mathbb{Z}_p[\theta]$;
for M Λ-module, let $M_\theta = M \otimes_\Lambda \Lambda_\theta$

Claim: If $\theta\omega^{-1}|_{(\mathbb{Z}/p\mathbb{Z})^\times} \neq 1$ or $\theta\omega^{-1}|_{(\mathbb{Z}/N\mathbb{Z})^\times} \neq 1$, then

$$Y_\theta \xrightarrow{\sim} \varprojlim_r H^2_{ét}(\mathcal{O}_r[\tfrac{1}{Np}], \mathbb{Z}_p(2))_\theta.$$

Corollary: ϖ induces $\varpi: S_\theta \to Y_\theta$ factoring through $P_\theta = (S/IS)_\theta$.

§4. Zeta elements and specialisations at infinity

For $(\alpha,\beta) \in \frac{1}{Np^r}\mathbb{Z}^2/\mathbb{Z}^2$, $\exists\, g_{\alpha,\beta} \in \mathcal{O}(Y(Np^r))^\times$ w/ q-expansion

$$g_{\alpha,\beta} = q^{\frac{1}{12}-\frac{\alpha}{2}+\frac{\alpha^2}{2}} \prod_{n=0}^{\infty}(1-q^{n+\alpha}e^{2\pi i\beta}) \prod_{n=1}^{\infty}(1-q^{n-\alpha}e^{-2\pi i\beta})$$

$$\in \mathcal{O}_r[\tfrac{1}{Np}][\![q^{1/12Np^r}]\!][q^{-1}]^\times.$$

[Crucial point]

The specialization of $g_{0,\frac{u}{Np^r}}$ at ∞ is the cyclotomic Np-unit $1-\zeta_{Np^r}^u$.

(Given by projecting to $\mathcal{O}_r[\frac{1}{Np}][\![q^{1/Np^r}]\!]$ and then evaluating at $q=0$).

Prop: $\exists$ a homomorphism of $\widetilde{\mathbb{T}}_r$-modules

$$z_r : M_r^0 \longrightarrow H^2_{ét}(Y_1(Np^r), \mathbb{Z}_p(2))$$

$$[u:v]_r \longmapsto g_{0,\frac{u}{Np^r}} \cup g_{0,\frac{v}{Np^r}}.$$

Aside. This is essentially how Kato constructs their classes.

Def.: (specialisation at ∞)

$$\infty_r : H^2_{ét}(Y_1(Np^r), \mathbb{Z}_p(2)) \to H^2_{ét}(\mathcal{O}_r[\tfrac{1}{Np}], \mathbb{Z}_p(2))$$

$$g_{0,\frac{u}{Np^r}} \cup g_{0,\frac{v}{Np^r}} \longmapsto (1-\zeta_{Np^r}^u, 1-\zeta_{Np^r}^v)$$

$$\varpi_r = \infty_r \circ z_r$$

Prop.: ∞_r is Eisenstein. This implies that ϖ_r is also Eisenstein.