MSRI ES Learning Sem.
3/10/2023

Odds and ends and Eisenstein cocycles

§1 ϖ

$N \geq 4$, p odd prime

C^0 nonzero cusps on $X_1(N)_{/\mathbb{C}}$ (i.e., $\not\mapsto 0 \in X_0(N)$).

$$S_N^0 = H_1(X_1(N), C^0, \mathbb{Z}_p)^+ \supset S_N = H_1(X_1(N), \mathbb{Z}_p)^+$$

$$Y_N^0 = (K_2(\mathbb{Z}[\mu_N, \tfrac{1}{N}]) \otimes \mathbb{Z}_p)^+ \cong H^2(\mathbb{Z}[\mu_N, \tfrac{1}{Np}], \mathbb{Z}_p(2))^+ \quad \text{(Steinberg} \mapsto \text{cup prod.)}$$

$$\cup\!\!|$$

$$Y_N = (K_2(\mathbb{Z}[\mu_N]) \otimes \mathbb{Z}_p)^+ \cong H^2(\mathbb{Z}[\mu_N, \tfrac{1}{p}], \mathbb{Z}_p(2))^+$$

$\Pi_N^0 : S_N^0 \to Y_N^0$, $\quad \Pi_N^0([u:v]^+) = \{1-\zeta_N^u, 1-\zeta_N^v\}^+$, $\quad u,v \in \mathbb{Z}/N\mathbb{Z} - \{0\}$, $(u,v)=(1)$

(Atkin-Lehner. Manin)

$\Pi_N : S_N \to Y_N$ restriction of Π_N

I Eisenstein ideal in any wt. 2, level N Hecke alg.

$I = (T_\ell - 1 - \ell\langle\ell\rangle,\ \ell \nmid N,\quad U_\ell - 1,\ \ell \mid N)$

Conj. (S.) (1) Π_N^0 is Eisenstein, $\Pi_N^0 \circ T = 0 \quad \forall T \in I$

(2) $\varpi_N : S_N/IS_N \to Y_N$ induced by Π_N is an isom.

Theorem (Fukaya-Kato) Π_N^0 is Eisenstein if $p \mid N$ or $p \nmid \varphi(N)$.

Theorem (S.-Venkatesh) $\Pi_N \circ (T_\ell - 1 - \ell\langle\ell\rangle) = 0 \quad \forall \ell \nmid N$.

$\Delta = (\mathbb{Z}/N\mathbb{Z})^\times = \Delta_p \times \Delta'$, Δ_p Sylow p-sgp. of Δ. $\quad \Lambda_N = \mathbb{Z}_p[\Delta]$.

Remark The obstruction in F-K proof lies in trivial Δ'-eigenspace. Is the conj. really true there?

§2 Υ

Now take $p \geq 5$ ($p=3$ should be fine). $\quad X_1(N)_{/\overline{\mathbb{Q}}}$.

$${}^0\mathcal{J}_N = H^1(X_1(N), \mathbb{Z}_p(1)) \quad \text{(étale cohom.)}$$

h_N = cuspidal $\mathbb{Z}_p$-Hecke alg. $\circlearrowright$ $^0\mathcal{J}_N$ via adjoint ops. (T_ℓ acts as T_ℓ^*).

$\theta : \Delta' \longrightarrow \overline{\mathbb{Q}}_p^\times$ even. $\mathbb{Z}_p(\theta)$ alg. of values.

For A Λ_N-mod., $A_\theta = A \otimes_{\Lambda_N} \mathbb{Z}_p(\theta)[\Delta_p]$ w/ $\Lambda_N \longrightarrow \mathbb{Z}_p(\theta)[\Delta_p]$ induced by θ.
$\Lambda_N \to h_N$ by $[j] \longmapsto \langle j \rangle^{-1}$ (meaning $[j]$ acts on $\mathcal{J}_N$ as $(\langle j\rangle^*)^{-1} = \langle j \rangle$).

Assumptions: 1) $N = Mp^r$, $p \nmid M\varphi(M)$ 2) θ primitive at $\ell \mid N$
3) $\theta\omega^{-1}(p) \neq 1$ for $\theta\omega^{-1}$ viewed as prim. Dir. char. 4) $\theta \neq \omega^2$.

$T_N = ({}^0\mathcal{J}_N / I\,\mathcal{J}_N)_\theta$. Under assumptions on θ, have SES
$0 \longrightarrow P_N \longrightarrow T_N \longrightarrow Q_N \to 0$ of $h_N[G_\mathbb{Q}]$-mod.
w/
- $P_N \cong (S_N / IS_N)_\theta$ as h_N-mod. w/ triv. $G_\mathbb{Q}$-action
- $Q_N \cong (\Lambda_N/\xi)^\iota_\theta(1)$ as $h_N[G_\mathbb{Q}]$-mod., ι = "σ_j acts as $[j]^{-1}$".
- seq. loc. split at all $\ell \mid Mp$.

Here, $\xi = \sum_{(j,N)=1} \frac{N}{2} B_2\left(\frac{j}{N}\right)[j]$. Mazur-Wiles $(h_N/I)_\theta \cong (\Lambda_N/\xi)_\theta$.

Remark) Assumptions (2) & (3) ensure SES splits at $\ell \mid M$ & p, resp.
Assumption (4) gives ξ_θ integral. (1) & (2) can at least be removed individually (upcoming thesis of F. Wu).

The map $T_N \to Q_N$ is given by pairing w/ Manin-Drinfeld splitting of $H^1(X_1(N), \mathbb{Q}_p(1)) \to H^1(Y_1(N), \mathbb{Q}_p(1))$ applied to $\xi\{0 \to \infty\}$, which lies in $\mathcal{J}_N$. (Pairing is twisted Poincaré duality).

This gives $G_\mathbb{Q} \to \mathrm{Hom}_h(Q_N, P_N) \cong P_N$ as h_N-mods. Restricts to homom. on $K = \mathbb{Q}(\mu_{Np^\infty})$ $\rightsquigarrow$ unram. Iw. mod. $X \longrightarrow P_N$ by local splitting
$(X(1)_{\mathrm{Gal}(K/\mathbb{Q}(\mu_{Mp}))})_\theta \cong (Y_N)_\theta$ & map factors through this.
$\rightsquigarrow$
$\Upsilon_{N,\theta} : (Y_N)_\theta \to (S_N/IS_N)_\theta$ homom. of Λ-mods.

§3 Conjecture
$\varpi_N \rightsquigarrow \varpi_{N,\theta}$ as well.

Conjecture (S.) $\Upsilon_{N,\theta}$ & $\varpi_{N,\theta}$ are inverse maps.

Theorem (Fukaya-Kato, FKS) $\xi_N' \Upsilon_{N,\theta} \circ \varpi_{N,\theta} = \xi_N'$.

Theorem (Ohta) $\Upsilon_{N,\theta}$ isom. if $\theta|_{(\mathbb{Z}/p\mathbb{Z})^\times}$ not injective.
(in $\varprojlim$ also have result of Wake & Wang-Erickson)

Remarks 1) $\Upsilon_{N,\theta}$ defined & equiv. form of conj. formulated long before $\varpi_{N,\theta}$.
2) ϖ_{Mp^r} & $\Upsilon_{Mp^r,\theta}$ trace/corer. compatible, so get conj. in $\varprojlim_r$
$X(1)_\theta \underset{\varpi}{\overset{\Upsilon}{\rightleftarrows}} (S_\Lambda/IS_\Lambda)_\theta$ $S_\Lambda = \Lambda$-adic mod. form.
3) Main conj. not needed for "$\mathfrak{h}/I \cong \Lambda/\xi$" : Emerton ($M=1$), Lafferty.
4) $\Upsilon_\theta \circ \varpi_\theta = 1$ refines Iwasawa main conj. (which follows from Υ_θ isom.), as it describes Υ_θ explicitly on symbols.
5) In 2022 preprint, I define a notion of "intermediate cohomology" to remove ξ' from both sides of F-K result. However, the two sides are no longer obviously connected. Obtain equivalent formulation requiring existence of "intermediate" zeta elts. modulo I.

§4 Cyclotomic units

Question: Does the ES of cyclotomic units also arise in the geometry of modular curves? I.e., are both divis. in IMC seen geometrically?

Yes, as described in Skinner's talk. He mentioned a long in progress joint project w/ Wake. I've never spoken on this, and it fits closely in w/ rest of talk, so I'd like to outline our perspective. I should note that essentially this construction was given by Anderson, C. Brinkmann (thesis), w/ extension by Harder, related work of Huber-Kings. We were motivated by a related cuspidal extn. class in work of Fukaya-Kato.

$N \geq 4$, $p \geq 5$ prime. C_∞ ∞-cusps in $X = X_1(N)$, C^0 non-∞-cusps.
$Y = Y_1(N) \subset Y_\infty = Y \cup C_\infty \subset X$. $\mathcal{Y}_1(N) = Y_1(N)/\mathbb{Z}[\frac{1}{Np}]$.
$\tilde{\mathcal{J}}_N = H^1_{C_\infty}(Y, \mathbb{Z}_p(1))$ étale cohom. compactly supp. at C_∞

$0 \to \tilde{H}^0(C_\infty, \mathbb{Z}_p(1)) \to \tilde{\mathcal{J}}_N \to H^1(Y_\infty, \mathbb{Z}_p(1)) \to 0$ SES
Here, $\tilde{H}^0(C_\infty) \cong \tilde{\Lambda}_N(1)$, $\tilde{\Lambda}_N = \Lambda_N/(\text{norm}, [-1])$.

For $(c,N)=1$, $\quad g = g_{0,\frac{1}{N}}^{(1-\sigma_c)(c^2-\sigma_c)} \in H^1(Y_\infty, \mathbb{Z}_p(1))^{G_\mathbb{Q}}$ via Kummer map.
Pull back by map from $\mathbb{Z}_p$, $1 \mapsto g$, to get
$0 \to \bar{\Lambda}_N^{a}(1) \to \mathcal{E}_N \to \mathbb{Z}_p \to 0.$
$\rightsquigarrow [\mathcal{E}_N] \in H^1(\mathbb{Z}[\frac{1}{Np}], \bar{\Lambda}_N^{a}(1)) \cong H^1(\mathbb{Z}[\mu_N, \frac{1}{Np}], \mathbb{Z}_p(1))^+ / H^1(\mathbb{Z}[\frac{1}{Np}], \mathbb{Z}_p(1))^+$.

__Theorem__ $[\mathcal{E}_N] = (1-\zeta_N)^{(1-\sigma_c)(c^2-\sigma_c)}$ (up to power of ζ_N).

__Proof__ (S.-Wake): Recall $\infty^*(g_{0,\frac{1}{N}}) = 1-\zeta_N$.
Spectral seq. gives comm. diagram $\quad \mathcal{Y}_\infty = \mathcal{Y}_1(N) \cup \infty\text{-cusps}$.

$$\begin{tikzcd} H^1(\mathcal{Y}_\infty, \mathbb{Z}_p(1)) \arrow[r, "\infty^*"] \arrow[d] & H^1(\mathbb{Z}[\mu_N, \frac{1}{Np}], \mathbb{Z}_p(1))^+ \arrow[d] \\ H^1(Y_\infty, \mathbb{Z}_p(1))^{G_\mathbb{Q}} \arrow[r, "\partial"] & H^1(\mathbb{Z}[\frac{1}{Np}], \bar{\Lambda}_N^{a}(1)) \end{tikzcd}$$

As $\partial(g) = [\mathcal{E}_N]$, done. //

__Question__ How to avoid use of g?

$$0 \to H^1(X, \mathbb{Z}_p(1)) \to H^1(Y_\infty, \mathbb{Z}_p(1)) \to \tilde{H}_0(C^\infty, \mathbb{Z}_p) \to 0$$
$$g \longmapsto \text{"}\mathrm{div}(g)\text{"}, \quad \zeta_N \text{ on } 0\text{-cusps}.$$

Manin-Drinfeld $\Rightarrow$ splits after $\otimes\, \mathbb{Q}_p$ as $h[G_\mathbb{Q}]$-mods.

Failure to split measured by congruence module, which is computed via residues of Eis. series (Ohta, Lafferty).
In fact, $d\log(g)$ is an Eis. series w/ res. $= \mathrm{div}(g)$
Given any elt. of $\tilde{H}_0(C^\infty, \mathbb{Z}_p)^{G_\mathbb{Q}}$ w/ integral splitting, can pull back to get extn. class. Computation of congr. module will tell us nonsplit in case of $\mathrm{div}(g)$.
Euler system relns. read off via Bernoulli dist.
If we project $\mathcal{E}_N$ to θ-eigenspace w/ θ even prim. $(\neq \omega^2)$, get $\mathrm{div}(g)_\theta = \mathrm{res}(G_\theta)$, $\quad G_\theta(q) = \sum_{n=1}^{\infty} \sum_{d \mid n} \theta^{-1}(\frac{n}{d})\, d\, q^n$, as in Skinner's talk.
So G_θ recovers $(1-\zeta_N)_\theta$.

§5 Eisenstein cocycles (joint w/ S-Venkatesh)

Toy case: $\mathbb{G}_m/\mathbb{Q}$. $\quad \mathbb{G}_m - \{1\} = \mathrm{Spec}\, \mathbb{Q}[z, \frac{1}{z}, \frac{1}{1-z}]$.

Motivic cohom.

$$\begin{array}{ccccccccc} 0 \to & H^1(\mathbb{G}_m, \mathbb{Z}(1)) & \longrightarrow & H^1(\mathbb{G}_m - \{1\}, \mathbb{Z}(1)) & \xrightarrow{\partial} & H^0(\{1\}, 1) & \to 0 \\ & \cong & & \cong & & \cong & \\ 0 \to & \mathcal{O}^\times_{\mathbb{G}_m} & \longrightarrow & \mathcal{O}^\times_{\mathbb{G}_m - \{1\}} & \xrightarrow{\partial} & \mathbb{Z} & \to 0 \\ & & & 1-z & \longmapsto & 1 & \end{array}$$

(simple root)

Here $\zeta_N \in \mathbb{G}_m(\mathbb{Q}(\mu_N))$ & $\zeta_N^*(1-z) = 1-\zeta_N$.

Idea: $\mathcal{E} \to X_1(N)$ univ. gen. e.c.

$$\begin{array}{ccc} \mathcal{E} - \{0\} & \leftrightarrow & \mathbb{G}_m - \{1\} \\ \uparrow \tau_N & & \uparrow \zeta_N \\ Y_1(N) & \underset{\infty}{\leftrightarrow} & \operatorname{Spec} \mathbb{Q}(\mu_N). \end{array}$$

Now consider $\mathbb{G}_m^2 / \mathbb{Q}$. $\Delta = M_2(\mathbb{Z}) \cap GL_2(\mathbb{Q})$ acts on right.

coord fns. z_1, z_2

Theorem (S.-Venkatesh) There is an "explicit" parabolic cocycle $\Theta : GL_2(\mathbb{Z}) \longrightarrow K_2(\mathbb{Q}(\mathbb{G}_m^2)) / \langle \{-z_1, z_2\} \rangle$ that is Eisenstein: $(T_\ell - \ell - [\ell]^*)\Theta$ is a coboundary $\forall$ primes ℓ.

Proof: Gersten complex $K = [K_2 \to K_1 \to K_0]$

$$K_2(\mathbb{Q}(\mathbb{G}_m^2)) \longrightarrow \bigoplus_{D \text{ divisor}} K_1(\mathbb{Q}(D)) \longrightarrow \bigoplus_{x \text{ zero cycle}} K_0(\mathbb{Q}(x))$$

$K_1(\mathbb{Q}(D)) \cong \mathbb{Q}(D)^\times$, $K_0(\mathbb{Q}(x)) \cong \mathbb{Z}$

K equivariant for left Δ-action by pullback.

Right exact w/ $H_2(K) \cong H^2(\mathbb{G}_m^2, \mathbb{Z}(2))$.

Take $e = [I] \in K_0^{GL_2(\mathbb{Z})}$. Connecting map $\rightsquigarrow [\Theta] \in H^1(GL_2(\mathbb{Z}), \overline{K}_2)$.

Choose $u = 1 - z_1^{-1}$ on $\mathbb{G}_m \times \{1\} \subseteq \mathbb{G}_m^2$, $\partial u = e$.

Can define $\Theta(\gamma)$ by $\partial \Theta(\gamma) = (\gamma^* - 1) u$.

Properties:
- u fixed by $\begin{pmatrix} 1 & 0 \\ * & \pm 1 \end{pmatrix} \Rightarrow \Theta$ parabolic.
- $\Theta(\gamma) =$ sum of $\gamma^* \{1 - z_1, 1 - z_2\} =: \langle \gamma \rangle$.

  ($\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \Rightarrow \exists k$ & $v_i = (b_i, d_i) \in \mathbb{Z}^2$ w/ $0 \le i \le k$ $v_0 = (0,1)$, $v_k = \det(\gamma)(b,d)$

  $\det\begin{pmatrix} b_{i-1} & b_i \\ d_{i-1} & d_i \end{pmatrix} = 1 \Rightarrow \Theta(\gamma) = \sum_{i=1}^{k} \langle \begin{pmatrix} b_i & -b_{i-1} \\ d_i & -d_{i-1} \end{pmatrix} \rangle$).
- $T_\ell e =$ sum of order ℓ subgps. of $\mu_\ell^2 = (\ell + [\ell]^*) e$

  $\Rightarrow (T_\ell - \ell - [\ell]^*)[\Theta] = 0$.

Ambiguity $H^2(\mathbb{G}_m^2, 2)$ reduced by considering $[m]_*$ fixed parts, $m \geq 1$. //

$K_2(\mathbb{Q}(\mathbb{G}_m^2)) \cong \varinjlim_{U} H^2(\mathbb{G}_m^2 - U, \mathbb{Z}(2))$ $\quad U \subseteq \mathbb{G}_m^2$ open subsch.

$\gamma \in \Gamma_0(N) \Rightarrow \Theta(\gamma)$ lies in lim over U w/ $(1, \zeta_n) \notin U$.

Define $\Theta_N = (1, \zeta_N)^* \Theta|_{\Gamma_0(N)}$.

<u>Theorem</u> (S.-V.) $\Theta_N : \Gamma_0(N) \to K_2(\mathbb{Q}(\mu_N)) / \langle \{-1, -\zeta_N\} \rangle$ is parabolic, Eisenstein for $\ell \nmid N$, and a homom. on $\Gamma_1(N)$ inducing Π.

(w/out inverting 2, get $\underset{\text{Manin}}{[u:v]} \mapsto \{1-\zeta_N^u, 1-\zeta_N^{-v}\}$).

<u>Cor</u> $\Pi \circ (T_\ell - 1 - \ell\langle \ell \rangle) = 0 \quad \forall \ell \nmid N$

<u>Remark</u> 1 Using $\mathcal{E}^2$ for $\mathcal{E} \to Y_1(N)$ universal, also get a Hecke-equivar. for $\ell \nmid N$ cocycle ${}_c\Theta : GL_2(\mathbb{Z}) \to K_2(\mathbb{Q}(\mathcal{E}^2)) \otimes \mathbb{Z}'$ that pulls back to Hecke equiv. for $\ell \nmid N$ zeta map $\qquad \mathbb{Z}' = \mathbb{Z}[\frac{1}{30}]$.

$z_N : H_1(X_1(N), \mathbb{Z}') \longrightarrow H^2(X_1(N), \mathbb{Z}'[\frac{1}{N}](2)) \qquad$ (or $\mathbb{Z}[\frac{1}{6}]$)

constructed by Goncharov & Brunault via other means (no Hecke).

Avoiding ω^2-eigensp. for $(\mathbb{Z}/p\mathbb{Z})^\times$, get Hecke for $\ell \nmid N$ zeta map

$z_N^{\mathrm{ord}} : H_1(X_1(N), \mathbb{Z}_p) \longrightarrow \underset{\text{idemp. removing } \omega^2}{\varepsilon} \cdot H^2(X_1(N), \mathbb{Z}_p(2))^{\mathrm{ord}}$

constructed by Fukaya-Kato.

Picture:
$$\begin{tikzcd}
\mathcal{E}^2 - \mathcal{E}^2[c] & \mathbb{G}_m^2 - \{1\} \arrow[l, dashed] \\
Y_1(N) \arrow[u, "{(0,\iota_N)}"', "\text{S-V}"] & \operatorname{Spec} \mathbb{Q}(\mu_N) \arrow[l, dashed, "\infty"', "\text{F-K}"] \arrow[u, "{(0,\zeta_N)}", "\text{S-V}"']
\end{tikzcd}$$

<u>Remark 2</u> The prespecialized Hecke-equivar. cohom. classes can be constructed very quickly via either a spectral seq. attached to K or via equivariant motivic cohom. (also suggested by Kings). E.g., $[\Theta]$ comes from $H^3_{GL_2(\mathbb{Z})}(\mathbb{G}_m^2 - \{1\}, \mathbb{Z}'(2)) \cong H^0_{GL_2(\mathbb{Z})}(\{1\}, \mathbb{Z}'(1)) \cong \mathbb{Z}'$.
(Parabolicity & specialization are more subtle)

Work-in-progress: $H_1(\text{Bianchi}) \longrightarrow H^2(\text{ray class field}/\text{imag. quad.})$ via Eisenstein cocycle for $E \times E'$
(Lecouturier, S., Shih, Wang) $\qquad E, E'$ CM by $\mathcal{O}_K$.