Berkeley_1

Christophe Cornut

April 12, 2023

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2. *There is one if and only if, for every place v of F,*

$$V \text{ does not split at } v \Longrightarrow E \text{ does not split at } v.$$

3. *If a (W, ψ) exists, then all embeddings*

$$U(W, \psi) \hookrightarrow SO(V, \varphi)$$

are conjugated under $SO(V, \varphi)$.

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The reciprocity map of E induces an isomorphism

$$T(\hat{F})/T(F) \xrightarrow{\simeq} \mathrm{Gal}(E[\infty]/E)$$

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We have a commutative diagram with exact rows and columns

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1 & \longrightarrow & \mu(E) & \longrightarrow & T(F) & \longrightarrow & \mathcal{P} & \longrightarrow & 1 \\
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When $H(F \otimes \mathbb{R})$ is not compact, this Galois group

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acts on

$$H^1(F)\backslash G(\widehat{F})/K$$

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Goal

We would like to understand

$$\mathrm{Gal}(E[\infty]/E) \quad \circlearrowright \quad \mathbb{Z}[H(F)\backslash G(\widehat{F})/K] \quad \circlearrowleft \quad \mathcal{H}_K$$

Motivation

- Let M be a motive over F which is
  irreducible, pure of weight -1, and symplectic of dimension $2n$,

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- Twist it by

  orthogonal Artin motives of dimension 2

  associated with ring class characters χ of E:

$$M \otimes N(\chi)$$

$$N(\chi) = \mathrm{Ind}_{E/F}\chi \qquad \chi : \mathrm{Gal}(E[\infty]/E) \to \overline{\mathbb{Q}}^{\times}$$

.

Motivation

Conjecture (Beilinson-Deligne-Bloch-Kato-Fontaine-Perrin-Riou)

There is an L-function with functional equation

$$L(M \otimes N(\chi), s) = \epsilon(M \otimes N(\chi), s) L(M \otimes N(\chi), -s).$$

Moreover,

$$\mathrm{ord}_{s=0} L(M \otimes N(\chi), s) = \dim H^1_{\mathrm{mot}}(F, M \otimes N(\chi))$$

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Corollary

The parity of $\dim H^1_f$ *is controlled by the root number* $\epsilon(M \otimes N(\chi))$.

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Conjecture (Rohrlich type)

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Construct it! Along the way, all choices should be governed by our single assumption on the root number, or cancel out.

Motives $\rightsquigarrow$ Automorphic Reps

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M corresponds to an algebraic automorphic representation Φ of GL_{2n}...

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Conjecture (Arthur?)

... a generic parameter of symplectic type for a Langlands-Vogan packet

$$\Pi(\Phi) = \{(G, \pi)\} / \sim$$

for automorphic cuspidal representations π of pure inner forms

$$G = SO(V) \qquad \dim_F V = 2n + 1, \quad \mathrm{disc}(V) = 1.$$

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- They give rise to Shimura varieties $\mathrm{Sh}(\mathbf{G}, \mathcal{X})$ where

$$\mathbf{G} = \mathrm{R}_{F/\mathbb{Q}} G \quad \text{and} \quad \mathcal{X} = \{\text{oriented negative } \mathbb{R}\text{-planes in } V_{v_0}\}.$$

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Automorphic Reps $\rightsquigarrow$ Shimura Varieties

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Fact (Langlands Conjecture / Milne-Shih)

The pull-back of $\mathrm{Sh}(\mathbf{G}, \mathcal{X})$ through $F \to \sigma_0 F$ does not depend on σ_0.

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Conjecture (Kottwitz?)

Let $\mathcal{V}$ be the corresponding local system. Then for any $\pi \in \Pi(G, \Phi)$,

$$H^\star(\mathrm{Sh}(\mathbf{G}, \mathcal{X}), \mathcal{V}(n))[\pi_f] = H^{2n-1}(\mathrm{Sh}(\mathbf{G}, \mathcal{X}), \mathcal{V}(n))[\pi_f] \simeq \sigma_{0,*}M$$

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Lemma

The map $g \mapsto \mathcal{Z}_K(g)$ gives a bijection

$$\mathbf{H}(\mathbb{Q}) \backslash \mathbf{G}(\mathbb{A}_f)/K \simeq \mathcal{Z}_K = \{\mathcal{Z}_K(g)\}.$$

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- An E-Hermitian F-hyperplane W of V gives a sub datum $(\mathbf{H}, \mathcal{Y})$ with

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- The reflex field is $\tilde{\sigma}_0 E$ where $\tilde{\sigma}_0|_F = \sigma_0$ and the dimension is $n-1$.
- For $g \in \mathbf{G}(\mathbb{A}_f) = G(\widehat{F})$, let $\mathcal{Z}_K(g)$ be the image of $g \times \mathcal{Y}$ in

$$\mathrm{Sh}_K(\mathbf{G}, \mathcal{X})(\mathbb{C}) = \mathbf{G}(\mathbb{Q}) \backslash \left(\mathbf{G}(\mathbb{A}_f)/K \times \mathcal{X} \right).$$

- This is an irreducible special cycle of codimension n defined over $E[\infty]$.

Lemma

The map $g \mapsto \mathcal{Z}_K(g)$ gives an $\mathcal{H}_K[\mathrm{Gal}(E[\infty]/E)]$-equivariant map

$$\mathbb{Z}[\mathbf{H}(\mathbb{Q}) \backslash \mathbf{G}(\mathbb{A}_f)/K] \to \mathrm{Cyc}^n(\mathrm{Sh}_K(\mathbf{G}, \mathcal{X})).$$

Goal

... and nice push-forward maps ...

- $\mathcal{Z}_K(g)$ is the image of the connected component $[\mathcal{Y}]$ through

$$\iota_g : \mathrm{Sh}_{\mathbf{H}(\mathbb{A}_f)\cap gKg^{-1}}(\mathbf{H},\mathcal{Y}) \to \mathrm{Sh}_{gKg^{-1}}(\mathbf{G},\mathcal{X}) \xrightarrow{g} \mathrm{Sh}_K(\mathbf{G},\mathcal{X})$$

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Goal

. . . giving motivic extensions . . .

There is an exact sequence

$$\begin{array}{ccccccccc} & & & & & & \mathbf{1} & & \\ & & & & & & \downarrow z_K(g) & & \\ 1 \to & H^{2n-1}(\mathrm{Sh}(\mathbf{G}), \mathcal{V}(n)) & \to & H^{2n-1}(\mathcal{Z}^c, \mathcal{V}(n)) & \to & H^{2n}_{\mathcal{Z}}(\mathrm{Sh}(\mathbf{G}), \mathcal{V}(n)) & \to & H^{2n}(\mathrm{Sh}(\mathbf{G}), \mathcal{V}(n)) \\ & \downarrow & & & & & & \\ & \pi_f^K \otimes M & & & & & & \end{array}$$

with $\mathcal{Z} = \mathcal{Z}_K(g)$.

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Abel-Jacobi

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with $\mathcal{Z}_K = \mathbf{H}(\mathbb{Q})\backslash\mathbf{G}(\mathbb{A}_f)/K$. By pull-back and push-out, we obtain

$$1 \to \pi_f^K \otimes M \to \star \to \mathcal{S}(\mathcal{Z}_K)_0 \to 1$$

$$\mathcal{S}(\mathcal{Z}_K)_0 = \ker\left(\mathcal{S}(\mathcal{Z}_K) \to H^{2n}(\mathrm{Sh}_K(\mathbf{G}), \mathcal{V}(n))\right)$$

An Euler System

Goal

. . . and an Euler System!

- In $\mathfrak{p}$-adic cohomology, we obtain an extension

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$$z_\mathfrak{p}(\phi) \in H^1(E[\phi], M_\mathfrak{p}).$$

- The distribution relations between these classes are encoded in the

$$\mathcal{H}_K[\mathrm{Gal}_E] - \text{structure of } \mathcal{S}(\mathcal{Z}_K) = \mathbb{Z}[H(F)\backslash G(\widehat{F})/K].$$

Choice of (G, π_f)?

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Conjecture (Arthur)

There are compatible bijections

$$\Pi(\Phi) \simeq \left(\prod_v \mathcal{S}_v / \mathcal{S}\right)^\vee \qquad \Pi(\Phi_v) \simeq \mathcal{S}_v^\vee$$

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Fact

$c^\chi \equiv c^E$ *is essentially independant of* χ.

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- Modify the Gross-Prasad c^E at ∞ to $c_i = c_f^E c_{i,\infty}$ with $c_{i,\infty}$ in

$$\left\{ (c_v)_{v|\infty} : \mathrm{sign}(c_v) = \begin{cases} (2n-1, 2) & v = v_0 \mid \infty \\ (2n+1, 0) & v_0 \neq v \mid \infty \end{cases} \right\}.$$

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Fact

This specific V indeed contains an E-hermitian F-hyperplane W.

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Local Setup

- Switch to local notations for F, E and

$$\begin{array}{ccccc} H^1 & \hookrightarrow & H & \overset{\det}{\twoheadrightarrow} & T \\ & & \downarrow & & \\ & & G & \hookleftarrow & K \end{array} \quad = \quad \begin{array}{ccccc} H^1(F_v) & \hookrightarrow & H(F_v) & \overset{\det}{\twoheadrightarrow} & T(F_v) \\ & & \downarrow & & \\ & & G(F_v) & \hookleftarrow & K_v \end{array}$$

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- We want to investigate the structure of

$$T \circlearrowright \mathbb{Z}[H^1 \backslash G / K] \circlearrowleft \mathcal{H}$$

What we want?

Suppose we are given:

$T_1 \subset T_0$: compact open subgroups of T,

o: an element of $H^1 \backslash G/K$ fixed by T_0,

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Question

Is there an $s \in \mathbb{Z}[H^1 \backslash G / K]$ fixed by T_1 such that

$$t \cdot o = \mathrm{Tr}_{T_0/T_1}(s)?$$

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$$t \cdot o = \mathrm{Tr}_{T_0/T_1}(s)?$$

There is such an s if and only if

$$\forall x \in H^1 \backslash G / K : \qquad [T_{0,x} : T_{1,x}] \mid n_x$$

where $T_{i,x}$ is the stabilizer of x in T_i and

$$t \cdot o = \sum n_x x.$$

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We need to compute

1. The support of $t \cdot o$
2. And for each x in this support,
   1. the stabilizer T_x of x in T
   2. the coefficient n_x of x in $t \cdot o$

Description of $H\backslash G/K$

We first describe the T-orbit space in $H^1\backslash G/K$, i.e.

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A toy case: linear groups

- V: finite free E-module of rank n and

$$H = GL_E(V) \quad \text{inside} \quad G = GL_F(V)$$

- K: hyperspecial in G, so

$$G/K = \{\mathcal{O}_F - \text{lattices in } V\}.$$

All $\mathcal{O}_F$-orders in E are Gorenstein

Definition (A chain of $\mathcal{O}_F$-orders)

$$\mathcal{O}_F \subset \cdots \subset \mathcal{O}_{c+1} \subset \mathcal{O}_c \subset \cdots \subset \mathcal{O}_1 \subset \mathcal{O}_0 = \mathcal{O}_E$$

$$\mathcal{O}_c := \mathcal{O}_F + \mathcal{P}_F^c \mathcal{O}_E \qquad c \in \mathbb{N}$$

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Fact

Each $\mathcal{O}_c$ is a local Gorenstein ring with maximal ideal

$$\mathcal{P}_c = \begin{cases} \mathcal{P}_E \subset \mathcal{O}_E & \text{if } c = 0, \\ \mathcal{P}_F \mathcal{O}_{c-1} & \text{if } c > 0 \end{cases}$$

unless $E = F \times F$ and $c = 0$, where $\mathcal{O}_0 = \mathcal{O}_E = \mathcal{O}_F \times \mathcal{O}_F$.

A result of Hyman Bass

Theorem (Hyman Bass)

For every $\mathcal{O}_F$-lattice L in V, there is an E-basis of V such that

$$L = \mathcal{O}_{c_1} e_1 \oplus \cdots \oplus \mathcal{O}_{c_n} e_n \quad \text{with} \quad c_1 \leq \cdots \leq c_n, \quad c_i \in \mathbb{N}.$$

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- c_n is the smallest integer c such that $\mathcal{O}_c L = L$
- c_{n+1-i} is the smallest integer c such that $\mathcal{O}_c \Lambda^i L = \Lambda^i L$

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$$L = \mathcal{O}_{c_1} e_1 \oplus \cdots \oplus \mathcal{O}_{c_n} e_n \quad \textit{with} \quad c_1 \leq \cdots \leq c_n, \quad c_i \in \mathbb{N}.$$

Remark

- c_n is the smallest integer c such that $\mathcal{O}_c L = L$
- c_{n+1-i} is the smallest integer c such that $\mathcal{O}_c \Lambda^i_E L = \Lambda^i_E L$

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Generalisation ($G = GL(V)$)

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For every F-norm α on V, there is an E-basis of V such that

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Corollary

This gives a bijection

$$\mathrm{inv} : H \backslash B(G) \simeq \mathcal{L}^{(n)}$$

where $\mathcal{L}^{(n)}$ is the set of "effective divisors" of degree n on

$$\mathcal{L} = E^\times \backslash \{F\text{-norms on } E\}.$$

What is $\mathcal{L}$?

From now on, we will assume that E/F is unramified.

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It takes $(e^{2i\pi\theta}, c)$ to the norm $q^{\theta} \|-\|_c : E \to \mathbb{R}_+$ with

$$\|z\|_c = q^{\frac{1}{2}c+k} \begin{cases} q^{-c} & \text{if } z \in \pi^{-k}(\mathcal{O}_n - \mathcal{P}_n) \\ q^{-\lceil c \rceil} & \text{if } z \in \pi^{-k}(\mathcal{P}_n - \pi\mathcal{O}_n) \end{cases} \qquad n = \lceil c \rceil$$

where

- q is the order of the residue field $\mathbb{F}$ of F
- $\pi\mathcal{O}_F = \mathcal{P}_F$

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Lemma

If $H_\alpha = H \cap G_\alpha$ is the stabilizer of α in H then

$$\det(H_\alpha) = \mathcal{O}^\times_{\lceil \min(c_i) \rceil} \quad \textit{if} \quad \mathrm{inv}(\alpha) = \sum_{i=1}^{n} (\star_i, c_i).$$

H\B(G) for G=SO(W)

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$$H = U(W) \quad \text{and} \quad G = SO(W)$$

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Then $B(G)$ is the set of *self-dual* norms α on W:

$$\alpha(x) = \alpha^*(x) \quad \text{with} \quad \alpha^*(x) = \sup\left\{\frac{|\varphi(x,y)|}{\alpha(y)} : y \in W \setminus \{0\}\right\}.$$

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Theorem

For every $\alpha \in B(G)$, there is a Witt E-decomposition

$$W = W_+ \oplus W_0 \oplus W_-$$

which is adapted to α. This means

$$\alpha(w_+ + w_0 + w_-) = \max(\alpha(w_+), \alpha(w_0), \alpha(w_-)\}.$$

Moreover, $\alpha(w_0) = |\varphi(w_0, w_0)|^{1/2}$ and

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This gives a bijection

$$\mathrm{inv} : H\backslash B(G) \simeq \overline{\mathcal{L}}^{(m)} \qquad \mathrm{inv}(\alpha) = \textit{class of}\ \mathrm{inv}(\alpha|W_+)$$

where $m = \dim_E W^+$ *is the Witt index of* W *and*

$$\overline{\mathcal{L}} = \mathrm{segment} \times \mathrm{half-line} = [-1, 1] \times \mathbb{R}_+.$$

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If $H_\alpha = H \cap G_\alpha$ is the stabilizer of α in H then

$$\det(H_\alpha) = \begin{cases} T_0 & \text{if } n > 2m \\ T_{\lceil \min(c_i) \rceil} & \text{in } n = 2m \end{cases} \quad \textit{where} \quad \mathrm{inv}(\alpha) = \sum_{i=1}^{n} (\star_i, c_i).$$

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- The bottom line gives equivariant embeddings

$$B(\underline{G}) \hookrightarrow B(G) \hookrightarrow B(\overline{G}).$$

H\B(G) for G=SO(V)

Consider the equivariant map

$$B(G) \to B(\underline{G}) \times B(\overline{G}) \qquad \alpha \mapsto (\underline{\alpha}, \overline{\alpha})$$

where $\underline{\alpha}$ and $\overline{\alpha}$ are the projection and extension of α.

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We obtain an injective invariant

$$\mathrm{inv} : H\backslash B(G) \hookrightarrow \overline{\mathcal{L}}^{(m)} \qquad \mathrm{inv}(\alpha) = \mathrm{inv}(\underline{\alpha}) + \mathrm{inv}(\overline{\alpha})$$

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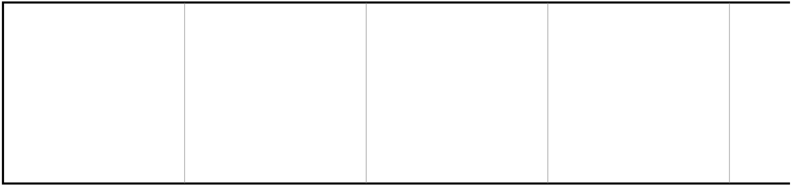
Working with inv (G split)

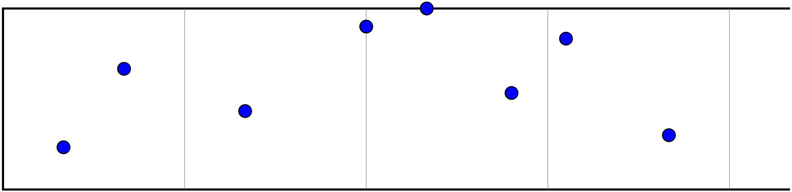
So $H\backslash B(G)$ is a subset of the set of

«effective divisors» of degree n on $[0,1]\times\mathbb{R}_+$.



Working with inv (G split)

Here is one such divisor, for $x \in B(G)$.



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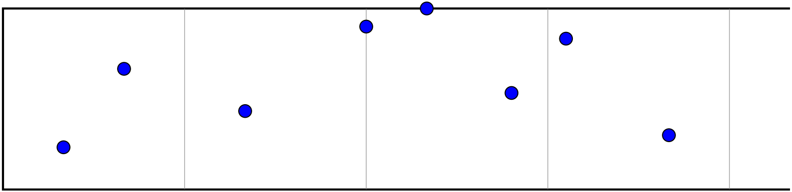
The stabilizer T_x of $[x] \in H^1 \backslash B(G)$ in T is given by:



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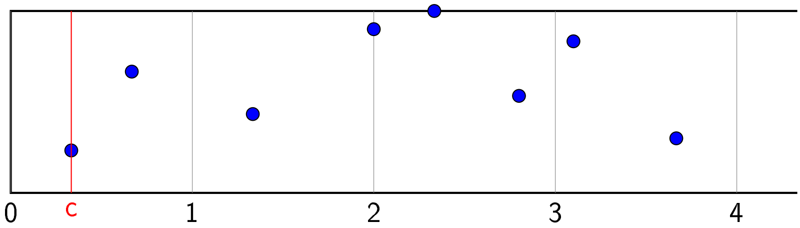

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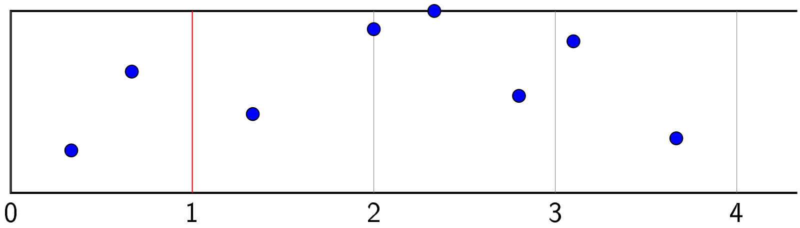
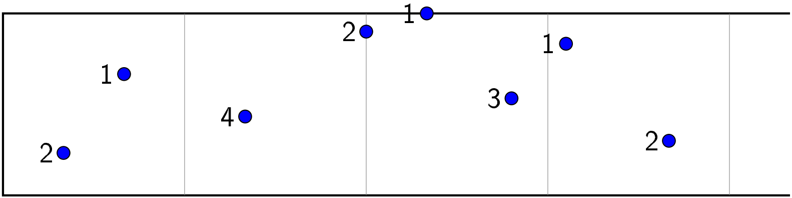


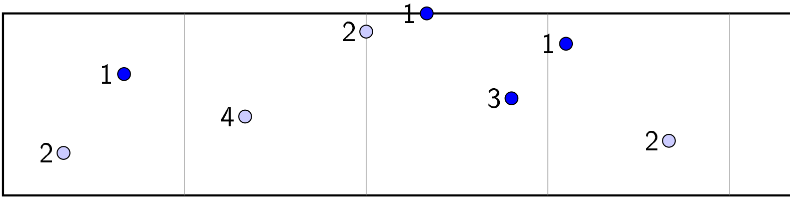
Working with inv (G split)

For this divisor to be in the image of inv ...

$$(n = 2 + 1 + 4 + 2 + 1 + 3 + 1 + 2 = 16)$$



. . . consider it modulo 2. . .



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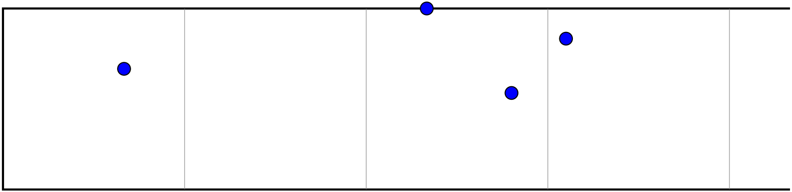
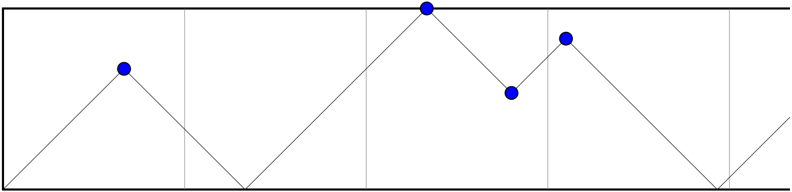


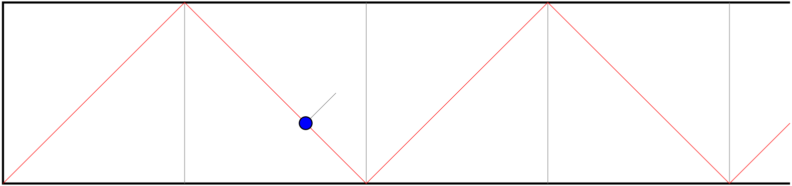
Working with inv (G split)

. . . then the remaining points have to be on a broken line:



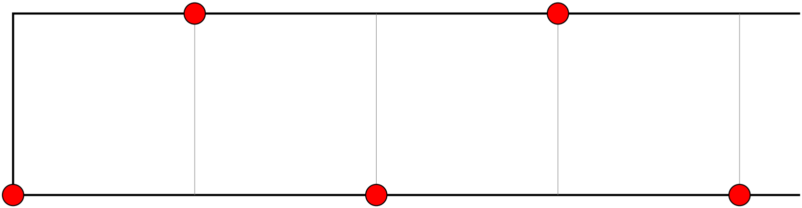
Working with inv (G split)

For $n = 1$: the point should just be on this broken line!



Working with inv (G split)

Here is the “support” of the G-hyperspecial H-orbits.



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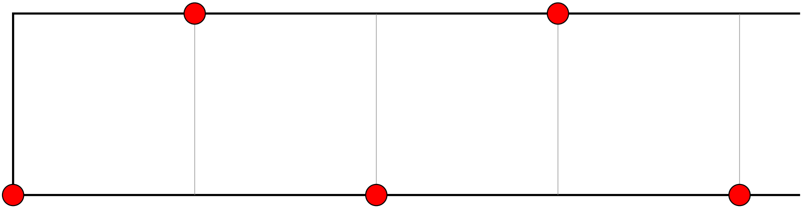
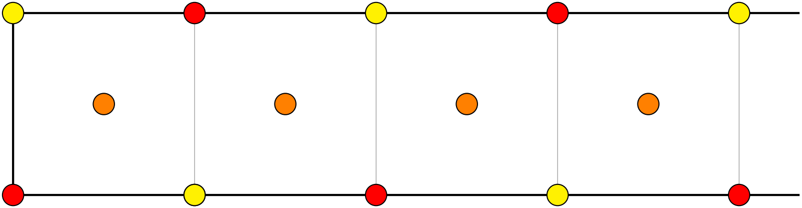
Here is the "support" of the G-hyperspecial H-orbits. We obtain:

$$H\backslash G/K = H\backslash B^{\circ}(G) \simeq \mathbb{N}^n_{\leq}$$



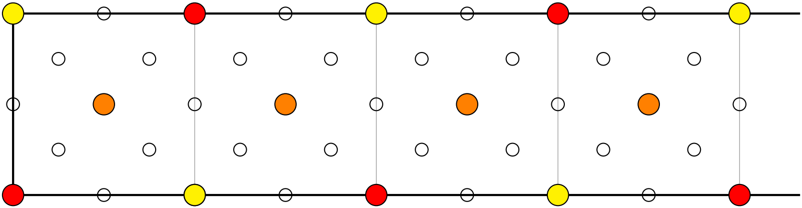
Working with inv (G split)

Here are the "support" of the H-orbits of *all* G-vertices...



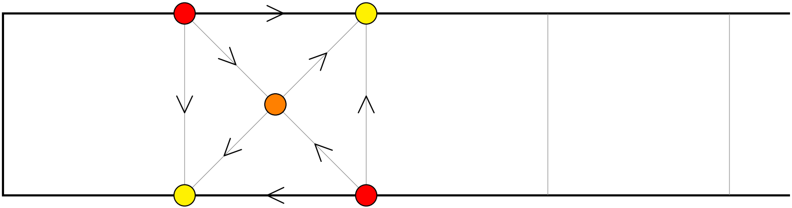
Working with inv (G split)

... and the “support” of H-orbits of mid-points of G-edges.



Working with inv (G split)

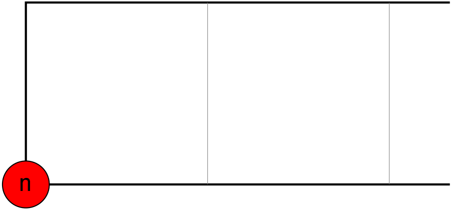
The source and target invariants are computed as follows.



Working with inv (G split)

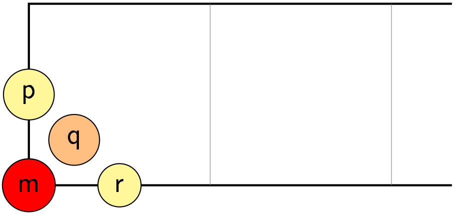
And the base point corresponds to an H-orbit of hyperspecials in $B(H)$:

$$B^{\circ}(H) = B(H) \cap B^{\circ}(G)$$



Working with inv (G split)

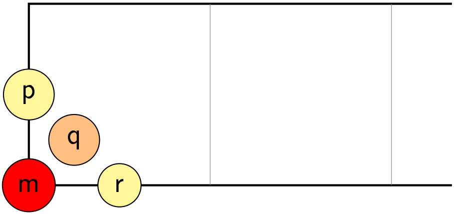
The orbits of the adjacent edges...



Working with inv (G split)

The orbits of the adjacent edges satisfy

$$m + p + q + r = n, \quad p \equiv (q-1)r \equiv 0 \bmod 2.$$



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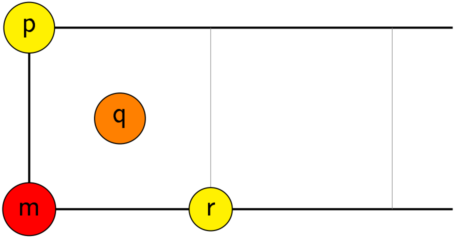
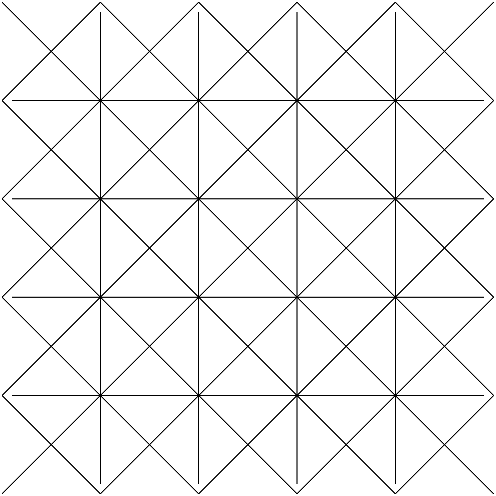


Vertices and oriented edges (split $SO(5)$)



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- An alcove
- A new point
- An half-alcove
- ... oriented!
- Neighbours of hyperspecial
- $\mathcal{T}_1$ operator
- $\mathcal{T}_2$ operator

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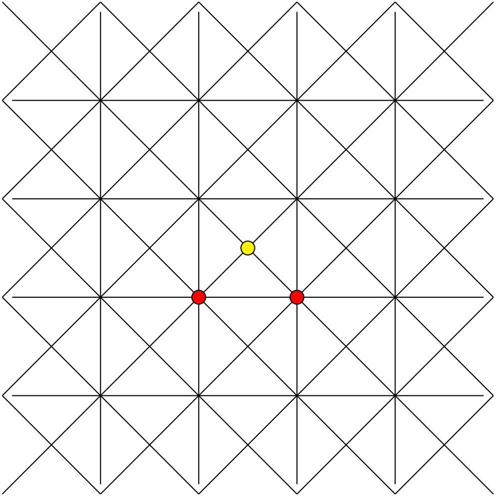

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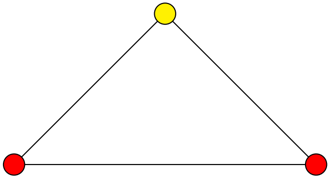

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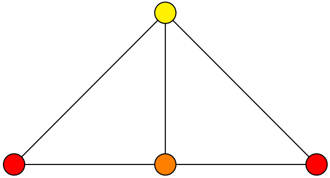

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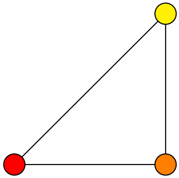

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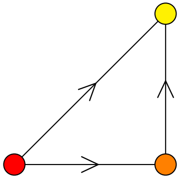
Vertices and oriented edges (split $SO(5)$)



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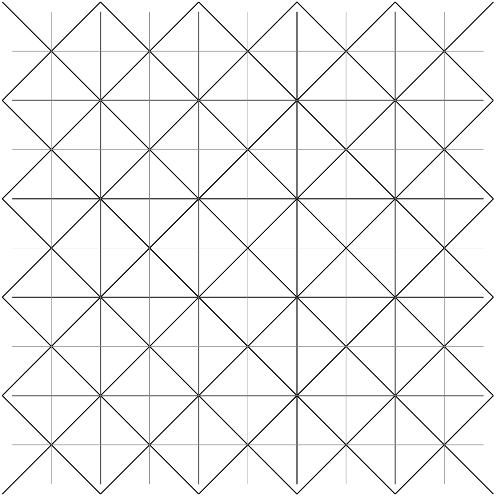

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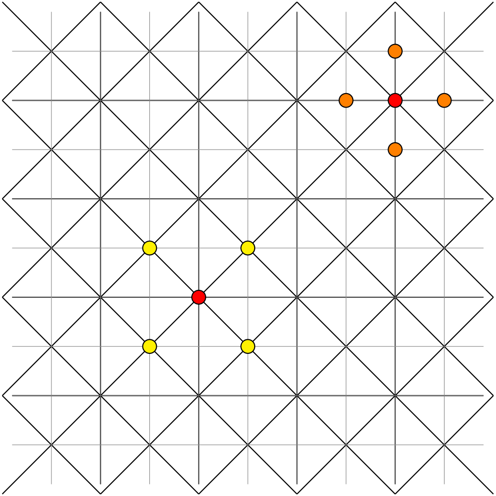

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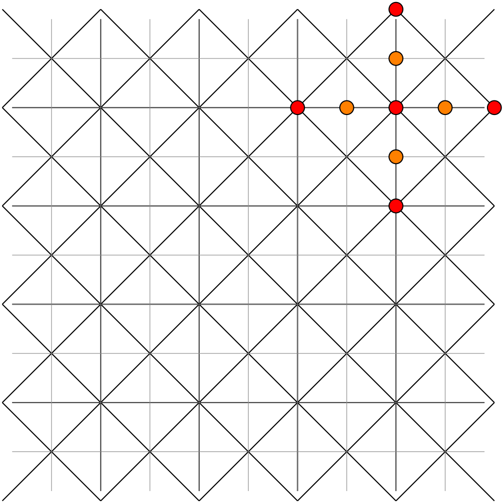

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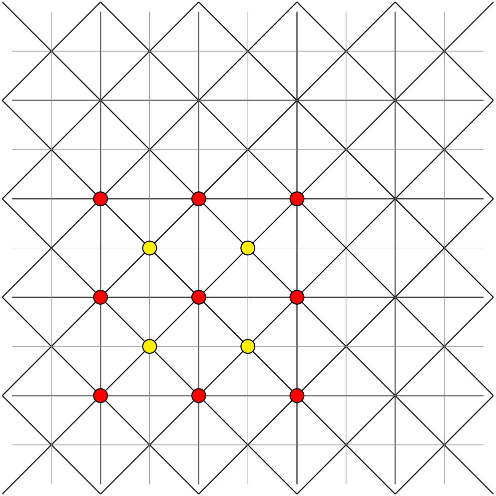

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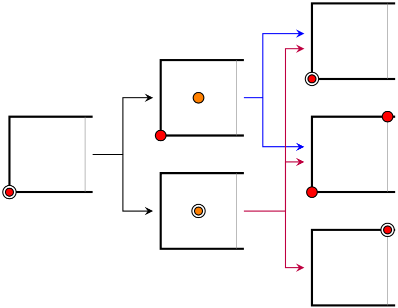

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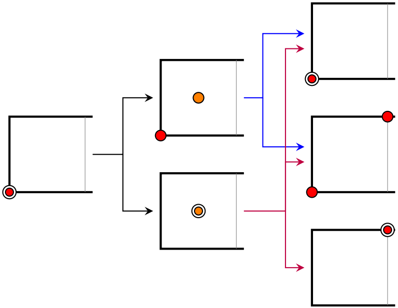

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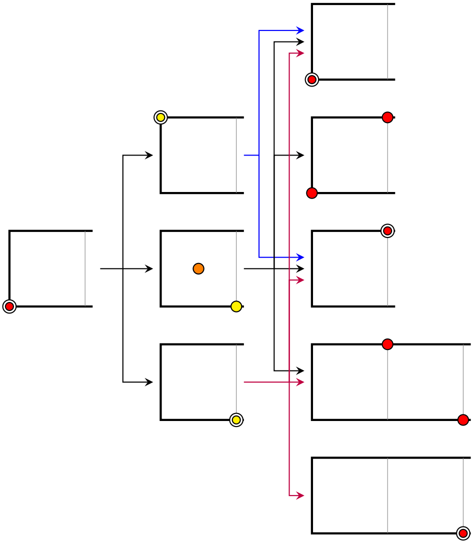

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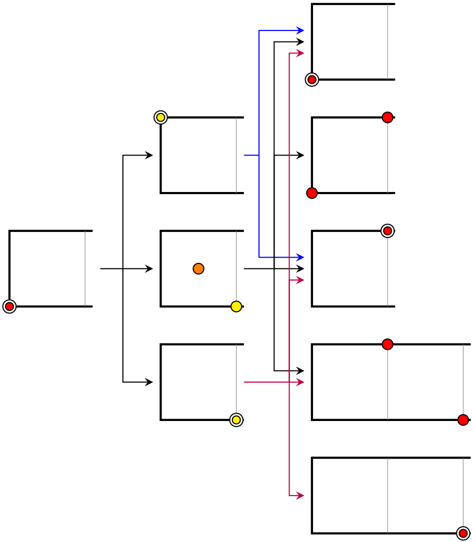

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Problem

There are TWO Hecke actions on

$$\mathbb{Z}[H^1\backslash G/K]$$

...and I mixed them up! Special thanks to **Waqar Ali Shah**!

The TWO Hecke actions on $\mathbb{Z}[H^1\backslash G/K]$

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- Any $\mathbb{Q}$-measure μ^1 on H^1 gives an isomorphism over $\mathbb{Q}$:

$$\mathbb{Q}[H^1\backslash G/K] + \text{bad action} \stackrel{\theta}{\to} \mathbb{Q}[H^1\backslash G/K] + \text{good action}$$

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where

$$\mu^1(x) = \mu^1(H^1 \cap gKg^{-1}) \quad \text{if} \quad x = H^1 gK.$$

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- The two actions have the same support.

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We want:

$$t_{\mathrm{good}}(o) = \sum n_x x, \qquad \forall x : \quad [T_{0,x} : T_{1,x}] \mid n_x.$$

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3. The projection $H^1\backslash G/K \to H\backslash G/K$ gives an equivariant map
$$\mathbb{Z}[H^1\backslash G/K] \twoheadrightarrow \mathbb{Z}[H\backslash G/K]$$
for the bad actions, which multiplies m_x by $[T_0 : T_x]$.

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We want: in $\mathbb{Z}[H\backslash G/K]$,

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where μ on H is normalized by $\mu(o) = 1$ and

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  - using the graph structure on vertices of $B(G)$,
  - viewing $B(G)$ as a space of norms for computations.

The method

A norm $\alpha \in B(G)$ has balls ($\mathcal{O}_F$-modules) and spheres ($\mathbb{F}$-vector spaces)

$$B(\alpha \leq q^\lambda) \qquad \text{and} \qquad S(\alpha, \lambda) = \frac{B(\alpha \leq q^\lambda)}{B(\alpha < q^\lambda)}$$

equipped with simple structures coming from G (=G-structures).

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Linear case: (1) H-induced structures on spheres.

- Let L be an $\mathcal{O}_F$-lattice in an E-vector space V. Then

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$$S^c = \ker\left(\frac{L}{\pi L} \to \frac{\mathcal{O}_c L}{\pi \mathcal{O}_c L}\right) = \frac{L \cap \pi \mathcal{O}_c L + \pi L}{\pi L}$$

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- Multiplication in V by $\eta \in \ker(\mathrm{Tr}_{E/F}) \cap \mathcal{O}_E^\times$ induces isomorphisms

$$\mathrm{Gr}^c(S) = S^c/S^{c-1} \xrightarrow{\simeq} \mathrm{Gr}_c(S) = S_{c-1}/S_c$$

and a structure of $\mathbb{E}$-vector space on $S(0) = S^0/S_0$.

Linear case: (2) Invariants.

Recall:

$$H\backslash G/K \simeq \mathbb{N}^n_{\leq} \quad \text{via} \quad L \simeq \mathcal{O}_{c_1} \oplus \cdots \oplus \mathcal{O}_{c_n}.$$

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Lemma

The multiplicity of c in $\mathrm{inv}(L) = (c_1, \cdots, c_n)$ is equal to

$$\begin{cases} \dim_{\mathbb{F}} \mathrm{Gr}_c S = \dim_{\mathbb{F}} \mathrm{Gr}^c S & \text{if } c \neq 0 \\ \dim_{\mathbb{E}} S(0) & \text{if } c = 0 \end{cases}$$

Linear case: (3) Edges

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$$\pi L_0 \subset L_1 \subset L_0 \quad \text{with} \quad \dim_{\mathbb{F}} L_0/L_1 = k.$$

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- Set $Gr(k,S) = k$-dimensional $\mathbb{F}$-spaces in $S = L/\pi L$. Thus

$$\left\{ L \xrightarrow{k} \star \right\} \xleftrightarrow{1:1} Gr(2n-k,S)$$

$$\left\{ \star \xrightarrow{k} L \right\} \xleftrightarrow{\quad 1:1 \quad} Gr(k,S)$$

Linear case: (4) Simple stratas

Edges $L_0 \xrightarrow{k} L_1$ between lattices with H-invariants

$$\left(\underline{\qquad}, \underbrace{c, \cdots, c}_{m} \right) \xrightarrow{k} \left(\underline{\qquad}, \underbrace{c, \cdots, c}_{m-k}, \underbrace{c+1, \cdots, c+1}_{k} \right)$$

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correspond to the following stratas:

- $S = L_1/\pi L_1$: singleton $\{S_c\}$ in $Gr(k, S)$.
- $S = L_0/\pi L_0$: big strata of W's in $Gr(2n-k, S)$ such that

$$m - k = \begin{cases} \dim_{\mathbb{E}} W_c \\ \dim_{\mathbb{F}} W_c \end{cases} \qquad W_c = \begin{cases} \text{largest } \mathbb{E}\text{-sub of } W & c = 0 \\ W \cap S_{c-1} & c > 0. \end{cases}$$

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correspond to the following stratas:

$S = L_1/\pi L_1$: singleton $\{S_c\}$ in $Gr(k, S)$.

$S = L_0/\pi L_0$: big strata of W's in $Gr(2n-k, S)$ such that

$$m - k = \begin{cases} \dim_{\mathbb{E}} W_c \\ \dim_{\mathbb{F}} W_c \end{cases} \qquad W_c = \begin{cases} \text{largest } \mathbb{E}\text{-sub of } W & c = 0 \\ W \cap S_{c-1} & c > 0. \end{cases}$$

Let $m_{1,0}$ $(= 1)$ and $m_{0,1}$ be the size of these strata.

Linear case: (5) Coefficients

Let $x_i \in H\backslash G/K$ correspond to L_i, so

$$x_0 = \left(\cdots, \underbrace{c, \cdots, c}_{m} \right) \xrightarrow{k} \left(\cdots, \underbrace{c, \cdots, c}_{m-k}, \underbrace{c+1, \cdots, c+1}_{k} \right) = x_1$$

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Fact

Set $t_k^{\pm} = K \begin{pmatrix} \pi^{\pm} I_k & \\ & I_{2n-k} \end{pmatrix} K \in \mathcal{H}$. Then

$$m_{1,0} \text{ is the coefficient of } x_0 \text{ in } t_k^{+}(x_1)$$

$$m_{0,1} \text{ is the coefficient of } x_1 \text{ in } t_k^{-}(x_0)$$

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Remark (on $m_{1,0} = 1$)

The *other* integer $m_{0,1}$ counts L_1's in $t_k^-(L_0)$ in a specified H-orbit. Since $L_0 = \mathcal{O}_c L_1$, we have $H_{L_1} \subset H_{L_0}$. So they form a single H_{L_0}-orbit, and

$$m_{0,1} = \frac{\mu(H_{L_0})}{\mu(H_{L_1})} = \frac{\mu(x_0)}{\mu(x_1)}.$$

Linear case: (5) Nice paths

Fix $c_1 \leq \cdots \leq c_n$ in $\mathbb{N}^n$. We want a path

$$\mathcal{O}_0 \oplus \cdots \oplus \mathcal{O}_0 \quad \rightsquigarrow \quad \mathcal{O}_{c_1} \oplus \cdots \oplus \mathcal{O}_{c_n}.$$

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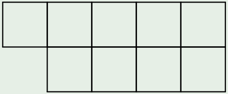
$$x_2 = \mathcal{O}_1 \oplus \mathcal{O}_2 \oplus \mathcal{O}_2 \oplus \mathcal{O}_2 \oplus \mathcal{O}_2$$



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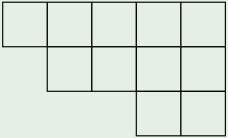
$$x_3 = \mathcal{O}_1 \oplus \mathcal{O}_2 \oplus \mathcal{O}_2 \oplus \color{red}{\mathcal{O}_3} \oplus \color{red}{\mathcal{O}_3}$$



Linear case: (5) Nice paths

Fix $c_1 \leq \cdots \leq c_n$ in $\mathbb{N}^n$. We want a path

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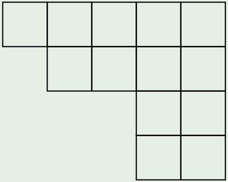

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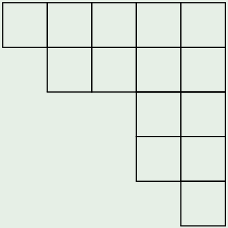
$$x_5 = \mathcal{O}_1 \oplus \mathcal{O}_2 \oplus \mathcal{O}_2 \oplus \mathcal{O}_4 \oplus \mathcal{O}_5$$



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$$x_0 \xrightarrow{5} x_1 \xrightarrow{4} x_2 \xrightarrow{2} x_3 \xrightarrow{2} x_4 \xrightarrow{1} x_5$$

So

$$\mu(x_5) = \frac{\mu(x_5)}{\mu(x_4)} \cdot \frac{\mu(x_4)}{\mu(x_3)} \cdot \frac{\mu(x_3)}{\mu(x_2)} \cdot \frac{\mu(x_2)}{\mu(x_1)} \cdot \frac{\mu(x_1)}{\mu(x_0)}$$

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- The H-structure is horrible when $G = SO(V)$.

This is the end...

Thank You!