New results on global bifurcation of traveling periodic water waves

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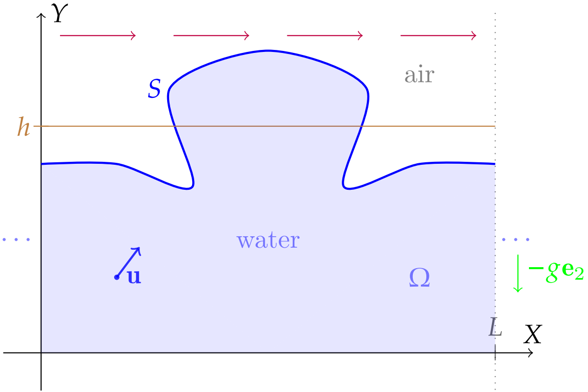
Joint work with Erik Wahlén

Sketch



- $2D$
- free boundary problem
- Ω water domain
- S free surface
- water inviscid, incompressible
- constant density
- Ω (and all appearing functions) L-periodic in X
- flat bed $Y = 0$
- presence of gravity and (possibly) surface tension $\rightsquigarrow$ constants $g > 0$, $\sigma \geq 0$

- Euler equations in Ω
- kinematic boundary condition on S and $Y = 0$
- dynamic boundary equation on S

Equations

Time-dependent

- Euler equation in $\Omega(t)$:

$$\frac{D\mathbf{u}}{Dt} := \mathbf{u}_t + (\mathbf{u}\cdot\nabla)\mathbf{u} = -\nabla(p + gY)$$

- incompressibility in $\Omega(t)$:

$$\nabla\cdot\mathbf{u} = 0$$

- kinematic boundary condition on $\partial\Omega(t) \overset{\text{loc.}}{=} \{F = 0\}$:

$$\frac{DF}{Dt} = 0$$

- dynamic boundary equation on $S(t)$:

$$p = p_{\text{atm}} - \sigma\kappa$$

(κ mean curvature)

constant speed c of propagation in $\mathbf{e}_1$-direction

$\longrightarrow\longrightarrow\longrightarrow$

change to moving frame

$\longrightarrow\longrightarrow\longrightarrow$

$X - ct \rightsquigarrow X$

$\mathbf{u} - c\mathbf{e}_1 \rightsquigarrow \mathbf{u}$

Steady

- Euler equation in Ω:

$$(\mathbf{u}\cdot\nabla)\mathbf{u} = -\nabla(p + gY)$$

- incompressibility in Ω:

$$\nabla\cdot\mathbf{u} = 0$$

- kinematic boundary condition on S and $Y = 0$:

$$\mathbf{u}\cdot\mathbf{n} = 0$$

- dynamic boundary equation on S:

$$p = p_{\text{atm}} - \sigma\kappa$$

Vorticity, stream function

- vorticity ω

$$\omega = \nabla \times \mathbf{u} = \partial_X \mathbf{u}_2 - \partial_Y \mathbf{u}_1$$

- vorticity equation (2D!)

$$(\mathbf{u} \cdot \nabla)\omega = 0 \tag{1}$$

- stream function ψ

$$\begin{aligned} \nabla \cdot \mathbf{u} = 0 \quad &\Rightarrow \quad \exists \psi : \mathbf{u} = \nabla^{\perp}\psi = (\psi_Y, -\psi_X) \\ &\Rightarrow \quad (\mathbf{u} \cdot \nabla)\psi = 0 \end{aligned} \tag{2}$$

- (in non-degenerate cases:) (1) and (2) imply a functional relation

$$\omega = \gamma(\psi)$$

- this gives

$$\Delta\psi = -\nabla \times \nabla^{\perp}\psi = -\nabla \times \mathbf{u} = -\omega = -\gamma(\psi)$$

Boundary conditions

- kinematic boundary condition:

$$\nabla^\perp \psi \cdot \mathbf{n} = \mathbf{u} \cdot \mathbf{n} = 0 \quad \Rightarrow \quad \psi \text{ locally constant on boundary}$$

- ψ determined up to constant $\rightsquigarrow$ consider

$$\psi = 0 \text{ on } S, \quad \psi = -m \text{ on } Y = 0 \; (m \in \mathbb{R})$$

- Bernoulli's law: hydraulic head

$$E = \frac{|\mathbf{u}|^2}{2} + p + gY + \int_0^\psi \gamma(s)\, ds = \frac{|\nabla\psi|^2}{2} + p + gY + \int_0^\psi \gamma(s)\, ds$$

constant in Ω:

$$\nabla E = (\mathbf{u} \cdot \nabla)\mathbf{u} - \omega \mathbf{u}^\perp + \nabla(p + gY) + \gamma(\psi)\nabla\psi = 0$$

- dynamic boundary condition:

$$p = p_{\text{atm}} - \sigma\kappa \quad \Rightarrow \quad \frac{|\nabla\psi|^2}{2} - \sigma\kappa + g(Y - h) = Q,$$

$$Q := E - p_{\text{atm}} - gh - \int_0^\psi \gamma(s)\, ds \text{ constant on } S$$

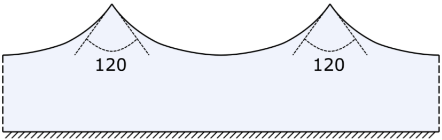
Summary

	original steady equations	stream formulation
incompressible Euler equation in Ω	$(\mathbf{u}\cdot\nabla)\mathbf{u} = -\nabla(p+gY)$ $\nabla\cdot\mathbf{u} = 0$	$\Delta\psi = -\gamma(\psi)$
kinematic boundary condition	$\mathbf{u}\cdot\mathbf{n} = 0$ on S and $Y = 0$	$\psi = 0$ on S $\psi = -m$ on $Y = 0$
dynamic boundary condition on S	$p = p_{atm} - \sigma\kappa$	$\frac{\lvert\nabla\psi\rvert^2}{2} - \sigma\kappa + g(Y-h) = Q$

History

Irrotational ($\gamma = 0$)

- Stokes mid 1800's: formal expansions, conjecture about wave of greatest height



- Levi-Civita, Struik, Nekrasov '20s, Jones & Toland '80s: small-amplitude waves
- Krasovskii '61, Keady & Norbury '78: large-amplitude waves
- Amick, Fraenkel & Toland '82, Plotnikov '82: proof of Stokes' conjecture

Rotational ($\gamma \neq 0$)

- Gerstner 1802, Crapper '57, Kinnersley '76: explicit solutions
- Dubreil-Jacotin '34: small-amplitude, small γ
- Goyon '54, Zeidler '73: general γ
- Constantin & Strauss '04: large amplitude, general γ
- '04–: Ambrose, Constantin, Ehrnström, Escher, Groves, Henry, Hur, Kozlov, Kuznetsov, Lokharu, Martin, Matioc, Matioc, Strauss, Varholm, Vărvărucă, Wahlén, Walsh, Weiss, Wheeler, Wright, . . .

Motivation

- ■ semi-hodograph transform $q = X$, $p = -\psi$
- ■ conformal change of variables
- ■ naive: scaling the height on each vertical ray

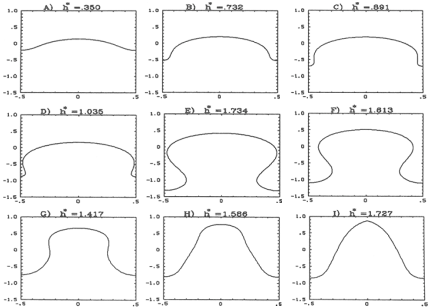
	surface tension	crit. layers/ stagn. pts.	vorticity	global bifurcation	overhanging profile
Constantin & Strauss '04 ■	✗	✗	general	Healey–Simpson degree	✗
Constantin & Vărvărucă '11 ■	✗	✓	constant	✗	✓
Martin '13 ■	✓	✓	constant	✗	✓
Constantin, Strauss & Vărvărucă '16 ■	✗	✓	constant	analytic	✓
Haziot & Wheeler '21 ■ [1]	✗	✓	constant	analytic	✓
Henry & Matioc '14 ■	✓	✓	monotone	identity + compact	✗
Varholm '20 ■	✗	✓	general	analytic	✗

Try to allow for all properties in full generality!
strategy: use conformal change of variables, rewrite equations as "identity plus compact"

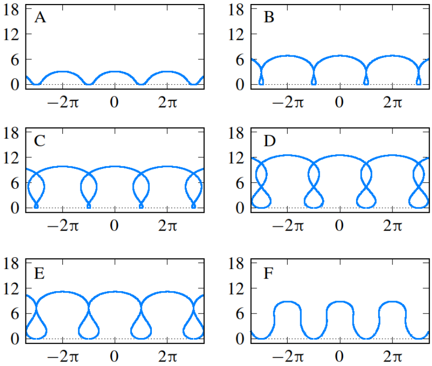
[1] solitary case

Do overhanging waves exist (without surface tension)?

- no, if flow is irrotational (Spielvogel '70; Amick '87)
- no, in case of constant vorticity for downstream flows (Constantin, Strauss & Vărvărucă '21)
- yes, numerical evidence in case of constant vorticity for upstream flows (e.g., Simmen & Saffman '85; Dyachenko & Hur '19)



Simmen & Saffman



Dyachenko & Hur

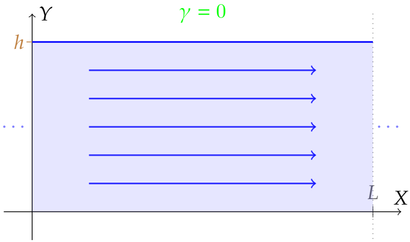
- yes, rigorous results for small gravity and constant vorticity (Hur & Wheeler '21)

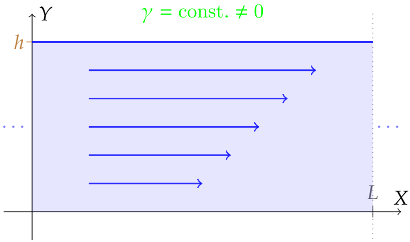
Why “identity plus compact”?

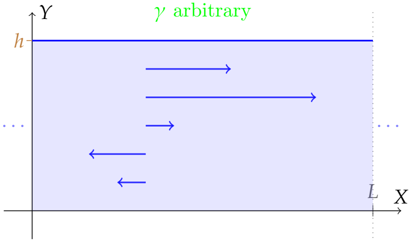
Or, how to obtain a global bifurcation result?

- degree methods especially useful in connection with semi-hodograph transform; requires assumptions (properness, Fredholmness, spectral properties) at all points, not only at solutions
- analytic methods: requires assumptions only at solutions, therefore easier to check; but analyticity is a strong condition
- identity plus compact: requires work to reformulate the equations, but saves some work later; is it even possible to reformulate the equations in this way?

Laminar flow solutions





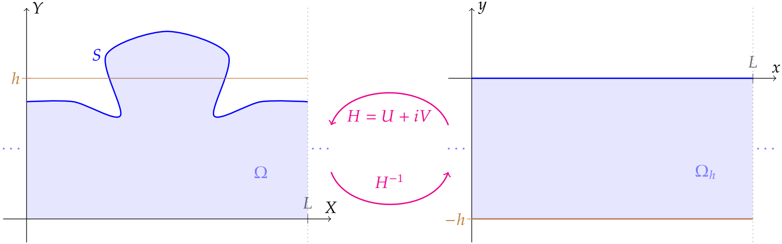


- we rather work with shifted $\psi^\lambda = \psi(\cdot + h)$
- equations for ψ^λ:

$$\psi^\lambda_{yy} = -\gamma(\psi^\lambda) \quad \text{on } [-h, 0],$$
$$\psi^\lambda(0) = 0, \qquad \psi^\lambda_y(0) = \lambda$$

- assume $\gamma \in C^{2,1}_{\text{loc}}(\mathbb{R})$, $\|\gamma'\|_\infty < \infty$
- $\lambda \in \mathbb{R}$ bifurcation parameter, $\lambda \equiv$ horizontal velocity at surface
- $m := m(\lambda) := -\psi^\lambda(-h)$

Conformal change of variables (see Constantin & Vărvărucă '11)



- $H = U + iV : \Omega_h \to \Omega$ conformal
- $H : \overline{\Omega_h} \to \overline{\Omega}$ homeomorphism
- "surface to surface, bottom to bottom"
- $U(x + L, y) = U(x, y) + L$,
  $V(x + L, y) = V(x, y)$
- h unique (fixed in the following)
- H unique up to translations in x
- S of class $C^{1,\beta}$, $\beta > 0$
  $\Rightarrow |dH/dz| = |\nabla V| \neq 0$ in $\overline{\Omega_h}$

- $V = V[w + h]$ uniquely determined by $w = V(\cdot, 0) - h$ via
$$\begin{aligned} \Delta V &= 0 && \text{in } \Omega_h, \\ V &= w + h && \text{on } y = 0, \\ V &= 0 && \text{on } y = -h \end{aligned}$$
- U harmonic conjugate of $-V$
- surface S parameterized by

$$S = \left\{ \left(x + (C_h^L w)(x), w(x) + h \right) : x \in \mathbb{R} \right\}$$

New equations

- C_h^L Fourier multiplier with symbol $-i\coth(k\nu h)$, $\nu := 2\pi/L$ ($\leftrightsquigarrow$ assume $\langle w\rangle = 0$)
- after suitable translation $U(x,0) = x + (C_h^L w)(x)$
- mean curvature

$$\kappa = \kappa[w](x) = \frac{\Big(1 + (C_h^L w')(x)\Big) w''(x) - w'(x)(C_h^L w'')(x)}{\Big((1 + (C_h^L w')(x))^2 + w'(x)^2\Big)^{3/2}}$$

- we work with the unknown (q, w, ϕ), where

$$\psi = (\phi + \psi^\lambda) \circ H^{-1}, \quad Q = \frac{\lambda^2}{2} + q$$

- natural assumptions: $w > -h$ on $\mathbb{R}$, $x \mapsto \Big(x + (C_h^L w)(x), w(x) + h\Big)$ injective on $\mathbb{R}$
- new equations:

$$\begin{aligned} \Delta\phi &= -\gamma(\phi + \psi^\lambda)|\nabla V|^2 + \gamma(\psi^\lambda) && \text{in } \Omega_h, \\ \phi &= 0 && \text{on } y = 0 \text{ and } y = -h, \end{aligned}$$

and $((\mathcal{S}f)(x) := f(x,0))$

$$\frac{(\mathcal{S}\phi_y + \lambda)^2}{2\Big((1 + C_h^L w')^2 + w'^2\Big)} - \sigma \frac{(1 + C_h^L w')w'' - w' C_h^L w''}{\Big((1 + C_h^L w')^2 + w'^2\Big)^{3/2}} + gw = Q \quad \text{on } \mathbb{R}$$

Identity plus compact

- focus now on the case $\sigma = 0$
- try to rewrite equations as
$$(q, w, \phi) = \mathcal{M}(\lambda, q, w, \phi)$$
with $\mathcal{M}$ compact
- easy: compactness in ϕ: $\mathcal{M}_3 := \mathcal{A}$, where $\mathcal{A} = \mathcal{A}(\lambda, w, \phi)$ solves
$$\begin{aligned} \Delta\mathcal{A} &= -\gamma(\phi + \psi^\lambda)|\nabla V|^2 + \gamma(\psi^\lambda) && \text{in } \Omega_h, \\ \mathcal{A} &= 0 && \text{on } y = 0 \text{ and } y = -h \end{aligned}$$
- harder: compactness in w

Compactness in w

Lemma

Under suitable regularity assumptions and if $\phi = \mathcal{A}(\lambda, w, \phi)$,

$$\frac{(\mathcal{S}\phi_y + \lambda)^2}{2\Big((1 + C_h^L w')^2 + w'^2\Big)} + gw = Q$$

is equivalent to

$$\Big\langle R\cos\Big((C_h^L)^{-1}\mathcal{P}(\ln R)\Big)\Big\rangle = 1,$$

$$w' = R\sin\Big((C_h^L)^{-1}\mathcal{P}(\ln R)\Big),$$

where

$$R(\lambda, q, w, \phi) := \frac{|\mathcal{S}\partial_y\mathcal{A}(\lambda, w, \phi) + \lambda|}{\sqrt{2(Q - gw)}}.$$

Rabinowitz

Theorem (Rabinowitz)

Assume

- X *Banach space,* $U \subset \mathbb{R} \times X$ *open,*
- $F \in C(U; X)$ *admits the form* $F(\lambda, x) = x + f(\lambda, x)$ *with* f *compact,*
- $F_x(\cdot, 0) \in C(\mathbb{R}; L(X, X))$,
- $F(\lambda_0, 0) = 0$,
- $F_x(\lambda, 0)$ *has an odd crossing number at* $\lambda = \lambda_0$ (satisfied if assumptions of Crandall–Rabinowitz are met).

Let $\mathcal{S}$ *denote the closure of the set of nontrivial solutions of* $F(\lambda, x) = 0$ *in* $\mathbb{R} \times X$ *and* $\mathcal{K}$ *denote the connected component of* $\mathcal{S}$ *to which* $(\lambda_0, 0)$ *belongs. Then one of the following alternatives occurs*:

(i) $\mathcal{K}$ *is unbounded;*

(ii) $\mathcal{K}$ *contains a point* $(\lambda_1, 0)$ *with* $\lambda_1 \neq \lambda_0$;

(iii) $\mathcal{K}$ *contains a point on the boundary of* U.

Main theorem

Theorem

Assume

- *there exists* $\lambda_0 \neq 0$ *such that*
  - 0 *is not in the Dirichlet spectrum of* $\partial_y^2 + \gamma'(\psi^{\lambda_0})$ *on* $[-h, 0]$,
  - *the dispersion relation* $d\big(-(k\nu)^2, \lambda_0\big) = 0$ *has exactly one solution* $k_0 \in \mathbb{N}$,
- *the transversality condition* $d_\lambda\big(-(k_0\nu)^2, \lambda_0\big) \neq 0$ *holds.*

Then one of the following alternatives occurs:

(i) $\mathcal{K}$ *is unbounded*: $|\lambda|$ *unbounded, or* w *unbounded in* $C^{0,\delta}_{\text{per}}(\mathbb{R})$ *for any* $\delta \in (5/6, 1]$, *or vorticity unbounded in* L^p *(physical domain) for any* $p > 1$;

(ii) $\mathcal{K}$ *contains a point* $(\lambda_1, 0, 0)$ *with* $\lambda_1 \neq \lambda_0$;

(iii) *a wave of greatest height is approached, i.e.,* $Q - g \max_{\mathbb{R}} w \to 0$ *along a sequence of solutions;*

(iv) *the conformal map degenerates, i.e.,* $\min_{\mathbb{R}}\Big((1 + C_h^L w')^2 + w'^2\Big) \to 0$ *along a sequence of solutions;*

(v) *self-intersection of the surface profile occurs;*

(vi) *intersection of the surface profile with the flat bed occurs.*

Remarks

- if $\sup \gamma' < \pi^2/h^2$, unboundedness of λ implies unboundedness of the relative mass flux $m(\lambda)$
- the norm in the unboundedness alternative for w is quite weak; in fact, this alternative can even be removed in case of downstream flows
- instead, $1/\min_{\mathbb{R}}\left((1+C_h^L w')^2 + w'^2\right)$ can be thought of as a "part of the norm of w" in an unboundedness alternative
- analytic global bifurcation (requiring Fredholmness at solutions and a certain compactness property of the solution set), which provides stronger conclusions, can also be immediately applied in case γ is real-analytic

Nodal properties

- maximum principles can be applied to the function $-\psi_X = \mathbf{u}_2$ and the linearized elliptic operator
$$-\Delta - \gamma'(\psi),$$
after changing to the flattened domain (similarly to Constantin, Strauss & Vărvărucă '16)
- however, some spectral assumption is needed; sufficient:
$$\sup \gamma' < \frac{\pi^2}{h^2}$$
- (typical) results:
  - no intersection with the flat bed
  - surface elevation strictly monotone from crest to trough
  - self-intersection of the surface can only happen exactly above a trough
  - looping back to a trivial solution cannot appear in many cases; e.g., if
$$\sup \gamma' < \frac{\pi^2}{4h^2} \quad \text{and} \quad \lambda_0 \gamma'' \geq 0$$

Thank you for your attention!