Regularity and Continuation for the Boltzmann Equation

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  - Known Results
  - Regularity Program
  - Carleman Decomposition

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Kinetic Theory

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Kinetic theory takes something of a “middle road”, operating on three scales: the *macroscopic* scale where things are largely observable, the *microscopic* encoding the interaction of particles, and the *mesoscopic* scale in between, which adds subtler structure while retaining the statistical nature.

Boltzmann Equation

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The behavior of a gas of particles interacting via local "collisions" is modeled by a time-varying distribution in space and velocity $f : \mathbb{R}_+ \times \mathbb{R}^3 \times \mathbb{R}^3 \to \mathbb{R}$ which satisfies

$$\begin{cases} \partial_t f + v \cdot \nabla_x f = Q(f,f) & \text{in } [0,\infty) \times \mathbb{R}^3 \times \mathbb{R}^3 \\ f|_{t=0} = f_0 & \text{in } \mathbb{R}^3 \times \mathbb{R}^3 \end{cases} \tag{B1}$$

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where f_0 is the initial distribution and

$$Q(f, g) := \int_{\mathbb{R}^d} \int_{\mathbb{S}^{d-1}} B(|v - w|, \theta) \left(f(w')g(v') - f(w)g(v) \right) d\sigma dw. \tag{B2}$$

B gives the statistical likelihood for collisions over the angles and relative velocities.

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The microscopic scale is seen in the interaction kernel Q, which belongs to a family that conserves kinetic energy and momentum after collision. For power-law forces, we have

$$B(r, \theta) = r^{\gamma} \theta^{-2-2s} b(\theta), \quad \gamma \in (-3, 1), \quad s \in (0, 1).$$

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The macroscopic scale is in the space part of the distribution. Notably in observable quantities like density, average momentum, kinetic energy, and entropy. The interaction is primarily through velocity transport.

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When f_0 is sufficiently close to the steady state $c_1 \exp(-c_2|v|^2)$, global well-posedness (and convergence to equilibrium) is known [Gressman-Strain 2011, He-Jiang 2017, Alonso-Morimoto-Sun-Yang 2018]. However, in weighted $L^\infty_{x,v}$, the previous state-of-the-art was [Silvestre 22], which only treated $\gamma + 2s \in [0, 2]$.

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The existence of near-vacuum global solutions was proved by [Chaturvedi 2021] (in a tenth-order Sobolev space with Gaussian weight), building off a similar result [Luk 2019] for the Landau equation.

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$$\text{Mass Density:} \quad M(t,x) := \int f(t,x,v)dv$$

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$$\text{Entropy Density:} \quad H(t,x) := \int f(t,x,v)\log(f(t,x,v))dv$$

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The long conditional regularity program of Imbert, Mouhot, and Silvestre has concluded that, in the physical regime $\gamma + 2s \in [0,2]$, solutions to (B1) exist, are unique, and are smooth *provided*:

$$m_0 \le M(t,x) \le M_0\ , \quad E(t,x) \le E_0\ , \quad H(t,x) \le H_0 \qquad \text{(CHQ)}$$

Carleman Decomposition

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For most of the analysis, we write $Q = Q_s + Q_{ns}$, where

$$Q_s(g, f) = \iint (f(v') - f(v))g(w')B \, d\sigma dw$$

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This can be rewritten as

$$Q_s(g, f) = \int K_g(v, v')(f(v') - f(v))dv'$$
$$K_g(v, v') = \frac{1}{|v - v'|^{3+2s}} \int_{(v'-v)^\perp} g(v + w)|w|^{\gamma+2s+1}\tilde{b}(\theta)dw$$
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The nonsingular term is generally well-behaved, and the singular term behaves almost like a fractional Laplacian (of order $2s$).

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Specifically, we need that for any $R > 0$, there exist $\lambda, \mu > 0$ such that for all $v \in B_R(0)$, there exists a symmetric $A(v) \subset \partial B_1(0)$ such that $|A| > \mu$ and $K_f(v, w) \geq \lambda |v - w|^{-3-2s}$ whenever $(v - w)/|v - w| \in A$.

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With a better (i.e. Gaussian) lower bound on f, one can get the cone without the entropy condition in (CHQ), yielding $\|f(t, \cdot, \cdot)\|_{L^\infty} \lesssim 1 + t^{-3/(2s)}$.

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Such a g is (R, δ, r)-*well-distributed* if for every $x \in \mathbb{R}^3$ there is $x_m \in B_R(x)$ and $v_m \in B_R(0)$ such that $g \geq \delta \chi_{B_r(x_m) \times B_r(v_m)}$.

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Theorem 1 (Henderson-Snelson-T. 2020)

Let $\gamma \in (-3, 1)$ and $s \in (0, 1)$. If f solves (B1) *with $M(t, x)$ and $E(t, x)$ bounded above for $t \in [0, T]$ and initially has mass, then*

$$f(t, x, v) \geq \mu(t, x)e^{-\eta(t,x)|v|^2}$$

with μ and η locally uniformly positive. If f is initially well-distributed, μ and η are independent of x.

New Continuation Criteria

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Theorem 2 (Henderson-Snelson-T. 2020)

Let f solve (B1) *and* $\gamma + 2s \in [0, 2]$ *on* $[0, T] \times \mathbb{T}^3 \times \mathbb{R}^3$.
(a) If f_0 is C^∞ and Schwartz in v and f has mass and energy density bounded above for $t \in [0, T]$, then

$$\|(1 + |v|^q) D^k f\|_{L^\infty([\epsilon, T])} \leq C_{q,k,\epsilon,T}$$

(b) If $\gamma < 0$ and $(1 + |v|)^q D^k f$ initially belongs to L^∞ for all q and k, the solution can be continued for as long as $M(t, x)$ and $E(t, x)$ remain bounded above.

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- Propagate lower bound to $(t_1, x_1, \frac{x_1 - x_0}{t_1})$ with similar barrier argument; some mass is transported along characteristics.

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For continuation, it is mostly a corollary, but the entropy bound is eliminated (when $\gamma < 0$) because our mass lower bound is robust enough to provide coercivity estimates for the collision operator (geometric kernel estimates).

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Theorem 3 (Henderson-Snelson-T. 2023)

For $\gamma \in (-3, 0)$, $s \in (0, 1)$, and $q > 2s + 3$, suppose $\langle v \rangle^q f_0 \in L^\infty(\mathbb{R}^6)$ and f_0 has mass. Then a local solution f to (B1) *exists for some time $T > 0$; f is locally to C_l^{2s} and agrees with f_0 at $t = 0$ in a weak sense. Better decay for f_0 implies greater classical regularity for f.*

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Theorem 4 (Henderson-Snelson-T. 2023)

As above, but with $q > \gamma + 2s + 3$, a weak solution f to (B1) exists for some time $T > 0$. If f_0 has mass, then f is locally Hölder continuous.

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A somewhat novel change of variables is also needed to adapt global regularity estimates.

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Theorem 5 (Henderson-Snelson-T. 2023)

For $\alpha > 0$ and $q > 0$ sufficiently large, suppose f_0 has mass at every point $x \in \mathbb{R}^3$, $\langle v \rangle^q f_0 \in L^\infty(\mathbb{R}^6)$, and $f_0 \in C_l^\alpha$. If f is the classical solution from Theorem 3 and g is any weak solution as in Theorem 4, then $f = g$.

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Implements a novel and non-kinetic propagation of Hölder bounds (in x and v), then passes them to t. Schauder estimates then give uniqueness via an energy method.

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- The exact rates are computed explicitly, and show that $\|f\|_{C_l^{2s+\alpha}} \lesssim t^{-\mu}\|f\|^{\nu}_{C_l^{\alpha'}}$ can only have $\mu < 1$ if $\alpha' > \alpha$. Even in principal, the De Giorgi and Schauder estimates should not suffice here.

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- If we integrate the equation for h, we can distribute derivatives between h and f. Since h and g are a priori weak, it has to be an integral in space and velocity. Known trilinear estimates for Q then yield

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- Since g lacks any form of lower bound, we can't use the $Q(g,h)$-term to close the estimate, and we need at least a uniform-in-x control on h and f in H_v^s. This is far from the known controlled quantities for (B1).

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$$\frac{d}{dt}\|h\|_{L^2}^2 \lesssim \int \|h\|_{L_v^2}\|f\|_{H_v^s}\|h\|_{H_v^s}dx + ...$$

- Since g lacks any form of lower bound, we can't use the $Q(g,h)$-term to close the estimate, and we need at least a uniform-in-x control on h and f in H_v^s. This is far from the known controlled quantities for (B1).
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- So we use an L^2-based energy estimate pared with a *propagated* control on a Hölder norm.
- The mass core condition guarantees a dynamic lower bound [Henderson-Snelson-T. 2020]: $f(t, x, v) \geq \mu e^{-\nu|v|^2}$ uniformly in t and x. This lets us use the strongest version of the Schauder estimates.

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Let $\mu(x, v) = Ce^{-|v|^2/2}$ *and* $q > 5$. *For initial data with* $\langle v \rangle^{\tilde{q}} f_0 \in L^\infty(\mathbb{R}^6)$, *there is* $\delta > 0$ *such that if* $\|\langle v \rangle^q (f_0 - \mu)\|_{L^\infty} < \delta$ *then the solution* f *to* (B1) *converges to* μ *as* $t \to \infty$ *faster than any polynomial rate.*

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Based on [Silvestre-Snelson, 2023], but Theorem 3 improves this in two ways: allows $\gamma + 2s < 0$ and allows finite polynomial decay in f_0. For the regime $\gamma + 2s < 0$, this is the first result showing convergence to equilibrium for perturbations in a zero-order space.

Thank You

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