Stability of Periodic Waves for NLS-Type Equations

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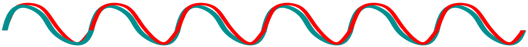
Stability of periodic waves

- **asymptotic behaviour of solutions for initial data close to a periodic wave**

Stability of periodic waves

- **asymptotic behaviour of solutions for initial data close to a periodic wave**
  - co-periodic perturbations [period T of the wave]



  - subharmonic perturbations [period NT, $N \in \mathbb{N}$]

  - **localized perturbations**

Dissipative PDEs

Nonlinear stability for localized/bounded perturbations:

- periodic wave trains in reaction-diffusion systems
- Taylor vortices in infinite cylinders
- viscous roll waves
- periodic traveling-wave solutions of viscous conservation laws
- ...

[Schneider; 1996-98] [Doelman, Sandstede, Scheel, Schneider; 2009] [Sandstede, Scheel, Schneider, Uecker; 2012] [Johnson, Noble, Rodrigues, Zumbrun; 2010-14] [...]

Dispersive PDEs

- **KdV, NLS equations and similar:**
  - nonlinear stability for co-periodic perturbations
  - spectral stability for localized/bounded/subharmonic perturbations
  - *use integrability:* orbital stability for subharmonic perturbations / linear stability for localized perturbations

[Gallay, H., Lombardi, Scheel, Angulo, Bona, Scialom, Bronski, Rapti, Deconinck, Kapitula, Pelinovski, Geyer, Hur, Johnson, Rodrigues, Natali, Pastor, . . . ; since 2005]

- Nonlinear stability for localized/bounded perturbations?

Defocusing NLS

$$\mathrm{i}U_t(x,t) + U_{xx}(x,t) - |U(x,t)|^2 U(x,t) = 0$$

- **orbital stability for co-periodic perturbations:** *use the general theory* **[Grillakis, Shatah, Strauss; 1990]***:*
  - a two-parameter family of periodic waves: [1]

$$U_{J,E}(x,t) = \mathrm{e}^{-\mathrm{i}t}\mathrm{e}^{\mathrm{i}px}\, Q_{J,E}(x)\,, \quad x,t \in \mathbb{R}$$

[1] J = angular momentum, E=energy in the steady equation

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  - *$Q_{J,E}$ is a degenerate saddle point of a modified energy with one unstable and two neutral directions*
  - conserved quantities: charge N and momentum M
  - *$Q_{J,E}$ is a local minimum of the modified energy restricted to the codimension two manifold*

    $$\Sigma_{J,E} \;=\; \left\{ Q \in X \;\middle|\; N(Q) = N(Q_{J,E})\,,\; M(Q) = M(Q_{J,E}) \right\}$$

[Gallay & H.; 2007]

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[1] J = angular momentum, E=energy in the steady equation

Defocusing NLS

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$2N\pi$–periodic perturbations:

- *similar analytical set-up* **but** *the second variation of the energy has* $2N-1$ *negative eigenvalues*
- *replace the manifolds* $\Sigma_{E,J}$ *by invariant manifolds of codimension* $2n \longrightarrow 2n$ *conserved quantities . . . ?*
- *integrability: take a higher order energy such that* $Q_{J,E}$ *is a local minimum* **[Gallay & Pelinovsky; 2015]**

Defocusing NLS

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- *integrability: take a higher order energy such that* $Q_{J,E}$ *is a local minimum* **[Gallay & Pelinovsky; 2015]**

localized perturbations:

- *the second variation of the energy has continuous spectrum ...?*
- How about a damped NLS equation?

LL EQUATION

[Lugiato & Lefever, 1987]

$$\frac{\partial \psi}{\partial t} = -i\beta \frac{\partial^2 \psi}{\partial x^2} - (1 + i\alpha)\psi + i\psi\,|\psi|^2 + F$$

- $\psi(x,t) \in \mathbb{C}$, $\beta, \alpha \in \mathbb{R}$, $F \in \mathbb{R}$ *(but not only)*
- *NLS-type equation with damping, detuning, and driving*
- *extensively studied in the physics literature* **[. . .]**
- *few mathematical results . . .*

Stability of periodic waves

Localized perturbations

- *spectral stability*

  [Delcey & H. (2018)]

- *spectral stability implies linear stability*

  [H., Johnson, Perkins (2021)]

- *linear stability implies nonlinear stability*

  [H., Johnson, Perkins, & de Rijk (2023)]

Spectral stability

- **spectrum of the linearized operator** $\mathcal{A}$ [matrix differential operator with periodic coefficients]

$$\mathcal{A} = -I + \mathcal{J}\mathcal{L}$$

$$\mathcal{J} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

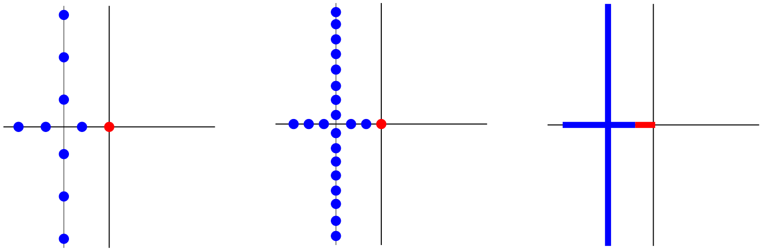
$$\mathcal{L} = \begin{pmatrix} -\beta\partial_x^2 - \alpha + 3\phi_r^2 + \phi_i^2 & 2\phi_r\phi_i \\ 2\phi_r\phi_i & -\beta\partial_x^2 - \alpha + \phi_r^2 + 3\phi_i^2 \end{pmatrix}$$

$\phi = \phi_r + i\phi_i$ denotes the T-periodic wave

Spectral stability

- **spectrum of the linearized operator** $\mathcal{A}$ [matrix differential operator with periodic coefficients]

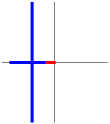
co-periodic — **subharmonic** — **localized**



space: $L^2_{\rm per}(0, NT)$ — space: $L^2_{\rm per}(0, NT)$ — space: $L^2(\mathbb{R})$

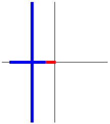
Localized perturbations

- **continuous spectrum**



Localized perturbations

- **continuous spectrum**



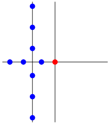
Key tool: **Bloch decomposition**

- **Bloch transform representation** for $g \in L^2(\mathbb{R})$

$$g(x) = \frac{1}{2\pi}\int_{-\pi/T}^{\pi/T} e^{i\xi x}\check{g}(\xi,x)d\xi, \quad \check{g}(\xi,x) := \sum_{\ell\in\mathbb{Z}} e^{2\pi i\ell x/T}\hat{g}(\xi+2\pi\ell/T)$$

- **Bloch operator** $\mathcal{A}_\xi := e^{-i\xi x}\mathcal{A}e^{i\xi x}$ acting in $L^2(0,T)$
- **spectrum**

$$\sigma_{L^2(\mathbb{R})}(\mathcal{A}) = \bigcup_{\xi\in[-\pi/T,\pi/T)} \sigma_{L^2_{\rm per}(0,T)}(\mathcal{A}_\xi)$$



Main result

Diffusive spectral stability

- *the spectrum of the linearized operator $\mathcal{A}$ acting in $L^2(\mathbb{R})$ satisfies*

$$\sigma_{L^2(\mathbb{R})}(\mathcal{A}) \subset \{\lambda \in \mathbb{C} : \mathrm{Re}(\lambda) < 0\} \cup \{0\};$$

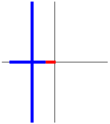
- *there exists $\theta > 0$ such that for any $\xi \in [-\pi/T, \pi/T)$ the real part of the spectrum of the Bloch operator $\mathcal{A}_\xi := e^{-i\xi x}\mathcal{A}e^{i\xi x}$ acting in $L^2_{\mathrm{per}}(0, T)$ satisfies*

$$\mathrm{Re}\left(\sigma_{L^2_{\mathrm{per}}(0,T)}(\mathcal{A}_\xi)\right) \leq -\theta\xi^2;$$

- *$\lambda = 0$ is a simple eigenvalue of $\mathcal{A}_0$ with associated eigenvector ψ (the derivative ϕ' of the periodic wave).*

- **decay of the C^0-semigroup $e^{\mathcal{A}t}$**

LINEAR STABILITY



- **decay of the C^0-semigroup $e^{\mathcal{A}t}$**
  - **difficulty:** no spectral gap
  - **Bloch decomposition of the semigroup**

$$e^{\mathcal{A}t}v(x) = \frac{1}{2\pi}\int_{-\pi/T}^{\pi/T} e^{i\xi x} e^{\mathcal{A}_\xi t}\check{v}(\xi,x)d\xi$$

Bloch operator $\mathcal{A}_\xi := e^{-i\xi x}\mathcal{A}e^{i\xi x}$ acting in $L^2_{\mathrm{per}}(0,T)$

[Schneider, . . . , Johnson, Noble, Rodrigues, Zumbrun]

Linear stability

Hypotheses

- **diffusive spectral stability;**
- the operator $\mathcal{A}$ generates a C^0-semigroup on $L^2(\mathbb{R})$ and for each $\xi \in [-\pi/T, \pi/T)$ the Bloch operators $\mathcal{A}_\xi$ generate C^0-semigroups on $L^2_{\mathrm{per}}(0, T)$;
- there exist positive constants μ_0 and C_0 such that for each $\xi \in [-\pi/T, \pi/T)$ the Bloch resolvent operators satisfy

$$\|(i\mu - \mathcal{A}_\xi)^{-1}\|_{\mathcal{L}(L^2_{\mathrm{per}}(0,T))} \leq C_0, \quad \text{for all } |\mu| > \mu_0.$$

checked for LLE: [Delcey, H., 2018], [Stanislavova, Stefanov, 2018]

Main result

- *There exists a constant* $C > 0$ *such that for any* $v \in L^1(\mathbb{R}) \cap L^2(\mathbb{R})$ *and all* $t > 0$ *we have* [2]

$$\left\| e^{\mathcal{A}t} v \right\|_{L^2(\mathbb{R})} \leq C(1+t)^{-1/4} \|v\|_{L^1(\mathbb{R}) \cap L^2(\mathbb{R})}.$$

- *Furthermore,* $e^{\mathcal{A}t} = s_p(t) + \widetilde{S}(t)$ *with*

$$\|s_p(t)v\|_{L^2(\mathbb{R})} \leq C(1+t)^{-1/4} \|v\|_{L^1(\mathbb{R})},$$

$$\left\| \widetilde{S}(t)v \right\|_{L^2(\mathbb{R})} \leq C(1+t)^{-3/4} \|v\|_{L^1(\mathbb{R}) \cap L^2(\mathbb{R})}.$$

[2] The decay is lost when $v \in L^2(\mathbb{R})$, only.

PROOF

estimates on Bloch semigroups $e^{\mathcal{A}_\xi t}$, $\xi \in [-\pi/T, \pi/T)$

(use: the diffusive spectral stability hypothesis, resolvent estimate, Gearhart-Prüss theorem)

- For any $\xi_0 \in (0, \pi/T)$, there exist $C_0 > 0$, $\eta_0 > 0$, such that

$$\left\| e^{\mathcal{A}_\xi t} \right\|_{\mathcal{L}(L^2_{\mathrm{per}}(0,T))} \leq C_0 e^{-\eta_0 t},$$

  for all $t \geq 0$ and all $\xi \in [-\pi/T, \pi/T)$ with $|\xi| > \xi_0$.
- There exists $\xi_1 \in (0, \pi/T)$ and $C_1 > 0$, $\eta_1 > 0$ such that

$$\left\| e^{\mathcal{A}_\xi t} (I - \Pi(\xi)) \right\|_{\mathcal{L}(L^2_{\mathrm{per}}(0,T))} \leq C_1 e^{-\eta_1 t},$$

  for all $t \geq 0$ and all $|\xi| < \xi_1$, where $\Pi(\xi)$ is the spectral projection onto the (one-dimensional) eigenspace associated to the eigenvalue $\lambda_c(\xi)$, the continuation for small ξ of the simple eigenvalue 0 of $\mathcal{A}_0$.

Proof

- **decompose the semigroup $e^{\mathcal{A}t}$** (use: the representation formula for the semigroup and a smooth cut-off function with $\rho(\xi) = 1$ for $|\xi| < \xi_1/2$ and $\rho(\xi) = 0$ for $|\xi| > \xi_1$)

$$
\begin{aligned}
e^{\mathcal{A}t} v(x) &= \frac{1}{2\pi} \int_{-\pi/T}^{\pi/T} \rho(\xi) e^{i\xi x} e^{\mathcal{A}_\xi t} \check{v}(\xi, x) d\xi \\
&+ \frac{1}{2\pi} \int_{-\pi/T}^{\pi/T} (1 - \rho(\xi)) e^{i\xi x} e^{\mathcal{A}_\xi t} \check{v}(\xi, x) d\xi \\
&=: S_{lf}(t) v(x) + S_{hf}(t) v(x)
\end{aligned}
$$

and show that

$$
\|S_{hf}(t) v\|_{L^2(\mathbb{R})} \lesssim e^{-\eta t} \|v\|_{L^2(\mathbb{R})}
$$

Proof

- **decompose** $S_{lf}(t)v(x)$ (use the diffusive spectral stability hypothesis)

$$
\begin{aligned}
S_{lf}(t)v(x) &= \frac{1}{2\pi}\int_{-\pi/T}^{\pi/T} \rho(\xi)e^{i\xi x}e^{\mathcal{A}_\xi t}\Pi(\xi)\check{v}(\xi,x)d\xi \\
&\quad + \frac{1}{2\pi}\int_{-\pi/T}^{\pi/T} \rho(\xi)e^{i\xi x}e^{\mathcal{A}_\xi t}(1-\Pi(\xi))\check{v}(\xi,x)d\xi \\
&=: S_c(t)v(x) + \widetilde{S}_{lf}(t)v(x)
\end{aligned}
$$

and show that

$$
\left\|\widetilde{S}_{hf}(t)v\right\|_{L^2(\mathbb{R})} \lesssim e^{-\eta t}\|v\|_{L^2(\mathbb{R})}
$$

Proof

decompose $S_c(t)v(x)$ (use formula for $\Pi(\xi)$)

$$S_c(t)v(x) = \frac{1}{2\pi}\int_{-\pi/T}^{\pi/T} \rho(\xi) e^{i\xi x} e^{\mathcal{A}_\xi t} \Pi(0) \check{v}(\xi,x) d\xi$$
$$+ \frac{1}{2\pi}\int_{-\pi/T}^{\pi/T} \rho(\xi) e^{i\xi x} e^{\mathcal{A}_\xi t} (\Pi(0) - \Pi(\xi)) \check{v}(\xi,x) d\xi$$
$$=: s_p(t)v(x) + \widetilde{S}_c(t)v(x)$$

and show that[3]

$$\left\| \widetilde{S}_c(t)v \right\|_{L^2(\mathbb{R})} \lesssim \|\xi e^{-d\xi^2 t}\|_{L^2_\xi(\mathbb{R})} \|v\|_{L^1(\mathbb{R})} \lesssim (1+t)^{-3/4} \|v\|_{L^1(\mathbb{R})}$$

$$\|s_p(t)v\|_{L^2(\mathbb{R})} \lesssim \|e^{-d\xi^2 t}\|_{L^2_\xi(\mathbb{R})} \|v\|_{L^1(\mathbb{R})} \lesssim (1+t)^{-1/4} \|v\|_{L^1(\mathbb{R})}$$

[3] The decay is lost when $v \in L^2(\mathbb{R})$, only.

Nonlinear stability

- **linear stability implies nonlinear stability**

Nonlinear stability

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  $\rightsquigarrow$ rely on Duhamel's formulation and properties of the semigroup

  $\rightsquigarrow$ **two main difficulties:**

  - semigroup with slow decay $(1+t)^{-1/4}$
  - C^0-semigroup

Nonlinear stability

First difficulty: semigroup with slow decay $(1+t)^{-1/4}$

- no decay for the (unmodulated) perturbation

$$\boxed{\tilde{v}(x,t) = \psi(x,t) - \phi(x)}$$

satisfying (Duhamel formulation)

$$\tilde{v}(t) = e^{\mathcal{A}t} v_0 + \int_0^t e^{\mathcal{A}(t-s)} \widetilde{\mathcal{N}}(\tilde{v}(s))\, ds$$

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- define a **modulated perturbation**

$$\boxed{v(x,t) = \psi(x - \gamma(x,t), t) - \phi(x)}$$

[Schneider, Doelman, Sandstede, Scheel, Uecker, . . . Johnson, Noble, Rodrigues, Zumbrun]

Nonlinear stability

modulated perturbation

$$v(x,t) = \psi(x - \gamma(x,t), t) - \phi(x)$$

$\rightsquigarrow$ satisfies $(\partial_t - \mathcal{A})(v + \gamma\phi') = \mathcal{N}(v, \gamma, \partial_t \gamma) + (\partial_t - \mathcal{A})(\gamma_x v)$

Nonlinear stability

modulated perturbation

$$\boldsymbol{v(x,t) = \psi(x-\gamma(x,t),t) - \phi(x)}$$

$\rightsquigarrow$ satisfies $(\partial_t - \mathcal{A})(v + \gamma\phi') = \mathcal{N}(v,\gamma,\partial_t\gamma) + (\partial_t - \mathcal{A})(\gamma_x v)$

$\rightsquigarrow$ use Duhamel formulation and $e^{\mathcal{A}t} = s_p(t) + \widetilde{S}(t)$ to:

- define the **phase modulation $\boldsymbol{\gamma(x,t)}$**

$$\boldsymbol{\gamma(t)} = s_p(t)v_0 + \int_0^t s_p(t-s)\mathcal{N}(v(s),\gamma(s),\partial_t\gamma(s))\,ds$$

(such that it captures the slowest decay rate $(1+t)^{-1/4}$)

- obtain a formula for $\boldsymbol{v(x,t)}$

$$\boldsymbol{v(t)} = \widetilde{S}(t)v_0 + \int_0^t \widetilde{S}(t-s)\mathcal{N}(v(s),\gamma(s),\partial_t\gamma(s))\,ds + \gamma_x(t)v(t)$$

(stronger decay rate $(1+t)^{-3/4}$; enough to conclude ...)

Second difficulty: C^0-semigroup

- no control of derivatives of the modulated perturbation

$$\boxed{v(x,t) = \psi(x-\gamma(x,t),t) - \phi(x)}$$

appearing in the nonlinear terms $\mathcal{N}(v(s),\gamma(s),\partial_t\gamma(s))$

Nonlinear stability

Second difficulty: C^0-semigroup

- no control of derivatives of the modulated perturbation

$$\boldsymbol{v(x,t) = \psi(x - \gamma(x,t), t) - \phi(x)}$$

appearing in the nonlinear terms $\mathcal{N}(v(s), \gamma(s), \partial_t \gamma(s))$

First approach

[H., Johnson, Perkins, & de Rijk (2023)]

- use **integration by parts** to gain derivatives and decay in the formula for the phase modulation $\boldsymbol{\gamma(x,t)}$

$$\boldsymbol{\gamma(t)} = s_p(t) v_0 + \int_0^t s_p(t-s) \mathcal{N}(v(s), \gamma(s), \partial_t \gamma(s))\, ds$$

Nonlinear stability

Second difficulty: C^0-semigroup

- no control of derivatives of the modulated perturbation

$$\boxed{v(x,t) = \psi(x-\gamma(x,t),t) - \phi(x)}$$

appearing in the nonlinear terms $\mathcal{N}(v(s), \gamma(s), \partial_t \gamma(s))$

First approach

[H., Johnson, Perkins, & de Rijk (2023)]

- use **integration by parts** to gain derivatives and decay in the formula for the phase modulation $\gamma(x,t)$

- also use the **unmodulated perturbation**

$$\boxed{\tilde{v}(x,t) = \psi(x,t) - \phi(x)}$$

(slow decay but no loss of derivatives)

Nonlinear stability

- for the unmodulated perturbation $\tilde{v}(x,t)$ and the modulated perturbation $v(x,t)$
  - obtain the decay rate $(1+t)^{-3/4}$ for the modulated perturbation [4]

$$\boxed{v(t) = \widetilde{S}(t)v_0 + \int_0^t \widetilde{S}(t-s)\mathcal{N}(v(s),\gamma(s),\partial_t\gamma(s))\,ds + \gamma_x(t)v(t)}$$

  - obtain the needed regularity for the unmodulated perturbation

$$\boxed{\tilde{v}(t) = e^{\mathcal{A}t}v_0 + \int_0^t e^{\mathcal{A}(t-s)}\widetilde{\mathcal{N}}(\tilde{v}(s))\,ds}$$

  - use **mean value inequalities** to connect $\tilde{v}(x,t)$ and $v(x,t)$

[4] Recall the decay rates in the decomposition $e^{\mathcal{A}t} = s_p(t) + \widetilde{S}(t)$

Nonlinear stability

Second approach

[Zumbrun (2023)]

- define a **forward-modulated perturbation**

$$\boxed{\mathring{v}(x,t) = \psi(x,t) - \phi(x + \gamma(x,t))}$$

- use the energy to obtain a nonlinear damping estimate for the forward-modulated perturbation
- use **mean value inequalities** to connect $\mathring{v}(x,t)$ and $v(x,t)$
- **Advantage:** *requires less regularity for the initial data (H^2 instead of H^4)*

Main result

- *There exist constants* $\varepsilon, M > 0$ *such that, whenever the initial perturbation* $v_0 \in L^1(\mathbb{R}) \cap H^4(\mathbb{R})$ *satisfies* $\boldsymbol{E_0 := \|v_0\|_{L^1 \cap H^4}} < \varepsilon$*, there exist functions*

$$\tilde{\boldsymbol{v}}, \gamma \in C\big([0,\infty), H^4(\mathbb{R})\big) \cap C^1\big([0,\infty), H^2(\mathbb{R})\big),$$

with $\tilde{v}(0) = v_0$ *and* $\gamma(0) = 0$ *such that* $\psi(t) = \phi + \tilde{v}(t)$ *is the unique global solution of LLE with initial condition* $\psi(0) = \phi + v_0$.

- *The inequalities*

$$\|\boldsymbol{\psi(t)} - \phi\|_{L^2},\ \|\boldsymbol{\gamma(t)}\|_{L^2} \leq \boldsymbol{ME_0(1+t)^{-\frac{1}{4}}},$$

$$\|\boldsymbol{\psi\left(\cdot - \gamma(\cdot, t), t\right)} - \phi\|_{L^2}, \leq \boldsymbol{ME_0(1+t)^{-\frac{3}{4}}},$$

hold for all $t \geq 0$.

A related problem

- **Subharmonic perturbations (NT-periodic):** stability results are not uniform in N
  - the size of initial data tends to 0 as $N \to \infty$
  - for LLE: the exponential decay rate tends to 0 as $N \to \infty$
- **Stability result uniform in N?**

A RELATED PROBLEM

- **Subharmonic perturbations (*NT*-periodic):** stability results are not uniform in N
  - the size of initial data tends to 0 as $N \to \infty$
  - for LLE: the exponential decay rate tends to 0 as $N \to \infty$

- **Stability result uniform in N?**

  Yes, *provided stability for localized perturbations holds:*
  - adapt the stability proofs used for localized perturbations
  - **the uniform decay rate is the same as the one for localized perturbations**
  - **improved nonuniform subharmonic stability result:** provides an N-independent ball of initial perturbations which eventually exhibit exponential decay at an N-dependent rate

Uniform subharmonic stability

- *There exist $\varepsilon, M > 0$ such that, for each $N \in \mathbb{N}$, whenever $v_0 \in H^2_{\mathrm{per}}(0, NT)$ satisfies $\boldsymbol{E_0 := \|v_0\|_{L^1 \cap H^2}} < \varepsilon$, there exist a constant $\boldsymbol{\sigma_{\mathrm{nl}}} \in \mathbb{R}$, a modulation function*

$$\boldsymbol{\gamma_{\mathrm{nl}}} \in C\big([0,\infty), H^4_{\mathrm{per}}(0,NT)\big) \cap C^1\big([0,\infty), H^2_{\mathrm{per}}(0,NT)\big),$$

and a global classical solution

$$\boldsymbol{\psi} \in C\big([0,\infty), H^2_{\mathrm{per}}(0,NT)\big) \cap C^1\big([0,\infty), L^2_{\mathrm{per}}(0,NT)\big),$$

of LLE with initial condition $\psi(0) = \phi + v_0$.

Uniform subharmonic stability

- *The inequalities*

$$\|\psi(\cdot,t)-\phi\|_{H^2_{\mathrm{per}}(0,NT)} \le ME_0,$$

$$\left\|\psi(\cdot,t)-\phi(\cdot+\frac{1}{N}\sigma_{\mathrm{nl}})\right\|_{L^2_{\mathrm{per}(0,NT)}} \le ME_0(1+t)^{-\frac{1}{4}},$$

$$\|\psi(\cdot,t)-\phi(\cdot+\gamma_{\mathrm{nl}}(\cdot,t))\|_{L^2_{\mathrm{per}(0,NT)}} \le ME_0(1+t)^{-\frac{3}{4}},$$

$$|\sigma_{\mathrm{nl}}| \le ME_0, \quad \left\|\gamma_{\mathrm{nl}}(\cdot,t)-\frac{1}{N}\sigma_{\mathrm{nl}}\right\|_{L^2_{\mathrm{per}(0,NT)}} \le ME_0(1+t)^{-\frac{1}{4}},$$

hold for all $t \ge 0$.

Corollary

- *For each* $N \in \mathbb{N}$*, there exists* $\delta_N > 0$ *such that for any* $\delta \in (0, \delta_N)$*, there exist constants* $T_\delta \geq 0$ *and* $M_\delta > 0$ *with*

$$\left\| \psi(\cdot, t) - \phi(\cdot + \frac{1}{N}\sigma_{\mathrm{nl}}) \right\|_{H^1} \leq \begin{cases} ME_0(1+t)^{-\frac{1}{4}}, & 0 < t \leq T_\delta, \\ M_\delta E_0 e^{-\delta t}, & t > T_\delta. \end{cases}$$

Furthermore, $T_\delta \to \infty$ *as* $N \to \infty$.

PROOF

Semigroup decomposition

$$\boxed{\mathrm{e}^{\mathcal{A}[\phi]t}v = \mathcal{P}_{0,N}v + \phi' s_{p,N}(t)v + \widetilde{S}_N(t)v,}$$

- constant term $\mathcal{P}_{0,N} = \phi'\langle\widetilde{\Phi}_0, \cdot\rangle_{L^2_N}/N$ *(spectral projection onto the one-dimensional kernel of $\mathcal{A}[\phi]$)*
- component with $(1+t)^{-1/4}$-decay
- component with $(1+t)^{-3/4}$-decay

Proof

Semigroup decomposition

$$\boxed{\mathrm{e}^{\mathcal{A}[\phi]t}v = \mathcal{P}_{0,N}v + \phi' s_{p,N}(t)v + \widetilde{S}_N(t)v,}$$

- constant term $\mathcal{P}_{0,N} = \phi'\langle\widetilde{\Phi}_0, \cdot\rangle_{L^2_N}/N$ *(spectral projection onto the one-dimensional kernel of $\mathcal{A}[\phi]$)*
- component with $(1+t)^{-1/4}$-decay
- component with $(1+t)^{-3/4}$-decay

Define the *inverse-modulated perturbation*

$$v(x,t) = \psi\left(x - \gamma_{\mathrm{nl}}(x,t), t\right) - \phi(x),$$

the *forward-modulated perturbation*

$$\mathring{v}(x,t) = \psi\left(x,t\right) - \phi\left(x + \gamma_{\mathrm{nl}}(x,t)\right).$$

and $\gamma_{\mathrm{nl}}(x,t) = \frac{1}{N}\sigma(t) + \gamma(x,t)$, with

$$\sigma(t) = \left\langle \widetilde{\Phi}_0, v_0 \right\rangle_{L^2_N} + \int_0^t \left\langle \widetilde{\Phi}_0, \mathcal{N}(v, \gamma, \partial_s\gamma, \partial_s\sigma)(s) \right\rangle_{L^2_N} \mathrm{d}s,$$

$$\gamma(t) = s_{p,N}(t)v_0 + \int_0^t s_{p,N}(t-s)\mathcal{N}(v,\gamma,\partial_s\gamma,\partial_s\sigma)(s)\mathrm{d}s.$$

PROOF

- Define the *template function:*

$$\eta(t) = \sup_{0\le s\le t}\Big[(1+s)^{\frac{3}{4}}\left(\|\mathring{v}(s)\|_{H^2_N} + \|\partial_x\gamma(s)\|_{H^3_N} + \|\partial_s\gamma(s)\|_{H^2_N}\right)$$
$$+ (1+s)^{\frac{1}{4}}\|\gamma(s)\|_{L^2_N} + (1+s)^{\frac{3}{2}}|\partial_s\sigma(s)| + |\sigma(s)|\Big]$$

(continuous, positive and monotonically increasing).

- Prove that there exist N- and t-independent constants $R > 0$ and $C \ge 1$ such that for all $t \in [0, \tau_{\max})$ if $\eta(t) \le R$ then

$$\eta(t) \le C\left(E_0 + \eta(t)^2\right)$$

and conclude by continuous induction, for sufficiently small E_0 and

$$\sigma_{\mathrm{nl}} = \left\langle \widetilde{\Phi}_0, v_0 \right\rangle_{L^2_N} + \int_0^\infty \left\langle \widetilde{\Phi}_0, \mathcal{N}(v, \gamma, \partial_s\gamma, \partial_s\sigma)(s) \right\rangle_{L^2_N} \mathrm{d}s.$$

Proof

Proof of $\eta(t) \le C\left(E_0 + \eta(t)^2\right)$:

- rely on connection between norms:

$$\|v(t)\|_{H^2_N} \le C\left(\|\mathring{v}(t)\|_{H^2_N} + \|\gamma_x(t)\|_{H^1_N}\right), \; \|\mathring{v}(t)\|_{L^2_N} \le C\left(\|v(t)\|_{L^2_N} + \|\gamma_x(t)\|_{H^1_N}\right)$$

- use Duhamel formulations to show that

$$\|v(s)\|_{L^2_N}, \; \|\partial_x^\ell \partial_s^j \gamma(s)\|_{L^2_N} \lesssim \frac{E_0 + \eta(s)^2}{(1+s)^{\frac{3}{4}}}, \; \|\gamma(s)\|_{L^2_N} \lesssim \frac{E_0 + \eta(s)^2}{(1+s)^{\frac{1}{4}}},$$

$$|\sigma(s)| \lesssim E_0 + \eta(s)^2, \quad |\partial_t \sigma(s)| \lesssim \frac{E_0 + \eta(s)^2}{(1+s)^{\frac{3}{2}}},$$

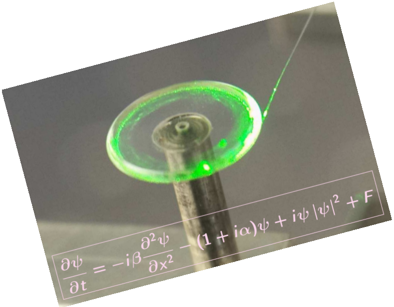
- use a nonlinear damping estimate

$$\|\mathring{v}(t)\|^2_{H^2_N} \lesssim \mathrm{e}^{-t}\|v_0\|^2_{H^2_N} + \|\mathring{v}(t)\|^2_{L^2_N} + \int_0^t \mathrm{e}^{-(t-s)}\left(\|\mathring{v}(s)\|^2_{L^2_N} + \|\gamma_x(s)\|^2_{H^3_N} + \|\partial_s\gamma(s)\|^2_{H^2_N} + |\partial_s\sigma(s)|^2\right)\mathrm{d}s$$

  and show that

$$\|\mathring{v}(s)\|_{H^2_N} \lesssim \frac{E_0 + \eta(s)^2}{(1+s)^{\frac{3}{2}}}$$

- combine these inequalites and conclude.


$\frac{\partial\psi}{\partial t} = -i\beta\frac{\partial^2\psi}{\partial x^2} - (1 + i\alpha)\psi + i\psi\,|\psi|^2 + F$