# Instantaneous gap loss of Sobolev regularity for the 2D incompressible Euler equations.

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# Generalized SQG equation $\alpha \in [0, 2]$

Let us consider a family of PDEs:

$$\begin{aligned} \partial_t \theta + u \cdot \nabla \theta &= 0 \\ u &= \nabla^{\perp} \psi \\ \theta &= -(-\Delta)^{1-\frac{\alpha}{2}} \psi, \end{aligned}$$

for $(x, t) \in \mathbb{R}^2 \times (0, \infty)$. Here $\alpha \in [0, 2]$.

# Local well-posedness in spaces of continuous functions

Gunther (1927), Lichtenstein (1930): LWP of the 2D Euler equations ($\alpha = 0$) for $u \in C^{k,\gamma}$, $k \geq 1$, $\gamma \in (0,1)$,
Constantin-Majda-Tabak (1994): connection of SQG ($\alpha = 1$) to $3D$ Euler,
Charney (1940s), Blumen (1978),
Held-Pierrehumbert-Garner-Swanson (1994) : geophysical origin of SQG,
Wu (2005): LWP in $C^{k,\gamma} \cap L^p$, $k \geq 1$, $\gamma \in (0,1)$, $p \geq 1$
Chae-Constantin-Wu (2011) : same for $\alpha \in [0,1]$,
Ambrose-Cozzi-Erickson-Kelliher (2022): dropped $L^p$ requirement for SQG ($\alpha = 1$),
Denisov (2015): double exponential growth of $\|\omega\|_{C^1}$ for a given finite time interval for 2D Euler ($\alpha = 0$),
Kiselev-Šverak (2014): double exponential growth of $\|\omega\|_{C^1}$ for **all times** for 2D Euler ($\alpha = 0$) in a disk,
Bourgain-Li (2015): strong ill-posedness for 2D Euler ($\alpha = 0$) for $u \in C^k$, $k \geq 1$,
(local well-posedness for $\theta \in C^k$, $k \geq 2$, $\alpha \in [0,1)$)

# Ill-posedness for SQG ($\alpha = 1$)

D. Córdoba and L. Martínez-Zoroa (2021):

### Theorem (Strong ill-posedness in $C^k$ for SQG)

*For any $\varepsilon > 0$, $k \geq 2$, $t_0 > 0$ there exists $\theta_0 \in H^{k+1/4} \cap C^k$ with $\|\theta_0\|_{C^k} \leq \varepsilon$ such that the unique solution $\theta(t) \in H^{k+1/4}$ satisfies*

$$\|\theta(t_0)\|_{C^k} \geq \frac{1}{\varepsilon}.$$

### Corollary (Non-existence in $C^k$ )

*Given $\varepsilon > 0$, $t_0 > 0$, $k \in \{2, 3, \ldots\}$, there exists $\theta_0 \in H^{k+1/8} \cap C^k$ with $\|\theta_0\|_{C^k} \leq \varepsilon$ and the unique solution $\theta(t) \in H^{k+1/8}$ satisfies*

$$\|\theta(t)\|_{C^k} = \infty \quad \text{for all } t \in (0, t_0].$$

# Local well-posedness in Sobolev spaces

Kato (1972), Kato-Ponce (1988): LWP for $u \in H^s$ for 2D Euler equations, $s > 1$,
Chae-Constantin-Córdoba-Gancedo-Wu (2012): LWP in $H^s$, $s > 1 + \alpha$, $\alpha \in [0, 2]$,
Chae-Wu (2012): LWP for a log-regularized SQG ($\alpha = 1$) in $H^2$,
Jolly-Kumar-Martinez (2022): extension to all $\alpha \in [0, 2]$,
Kiselev-Nazarov (2012): norm inflation in $H^s$ ($s \geq 11$) in a long time for SQG ($\alpha = 1$),
Kiselev-Yao-Zlatoš-Ryzhik (2016): LWP and **finite time blow-up** for patches in $\mathbb{R}^2_+$ (low regularity),
Bourgain-Li (2015): Strong ill-posedness of 2D Euler ($\alpha = 0$) for $\theta \in H^1$,
Elgindi-Jeong (2017): Different proof,
Jeong (2021): Strong ill-posedness $\theta \in W^{1,p}$ for 2D Euler equations in the Yudovich class $\theta \in L^\infty$, for some $p \in (1, 2)$, such that $\theta(t)$ loses $W^{1,p(t)}$ regularity continuously in time $t$.

# Instantaneous gap loss for 2D Euler ($\alpha = 0$)

D. Córdoba, L. Martínez-Zoroa, W. O. (2022):

## Theorem (Sobolev norm inflation)

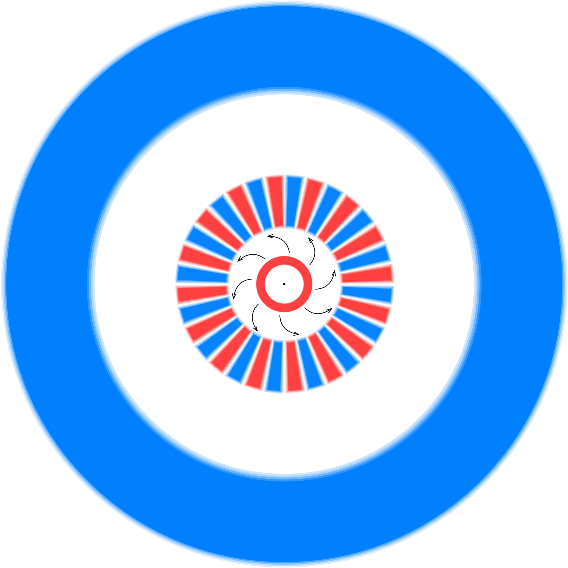
*For every $\varepsilon > 0$, $T > 0$, $\beta \in (0,1)$, $\beta' > (2-\beta)\beta/(2-\beta^2)$, there exists initial vorticity $\omega_0 \in C_0^\infty(\mathbb{R}^2)$ such that $\|\omega_0\|_{H^\beta} \leq \varepsilon$ and the unique solution $\omega$ to the 2D Euler equations satisfies*

$$\|\omega(t)\|_{H^{\beta'}} \geq \frac{1}{\varepsilon} \qquad \text{for all } t \in [1/T, T].$$

## Corollary (Gap loss of Sobolev regularity)

*For every $\beta \in (0,1)$, there exists initial vorticity $\omega_0 \in H^\beta(\mathbb{R}^2)$ such that $\|\omega_0\|_{H^\beta} \leq \varepsilon$ and the unique solution $\omega(t)$ to the 2D Euler equations satisfies* **does not belong** *to $H^{\beta'}(\mathbb{R}^2)$* **for all** *$\beta' > (2-\beta)\beta/(2-\beta^2)$ and all $t > 0$.*

Sobolev norm inflation for the 2D Euler equations - Sketch of the initial vorticity $\omega(0) = \omega_{\mathrm{rad}}(0) + \omega_{\mathrm{osc}}(0)$.



Thank you for your attention.