Finite-time blowup for an Euler and hypodissipative Navier–Stokes model equation on a restricted constraint space

Evan Miller

University of British Columbia
Pacific Institute for the Mathematical Sciences

Introduction

- The incompressible Euler and Navier–Stokes equations are among the oldest PDEs, but core questions about their solutions in three dimensions remain open.

$$\partial_t u + \mathbb{P}((u \cdot \nabla)u) = 0$$
$$\partial_t u - \nu\Delta u + \mathbb{P}((u \cdot \nabla)u) = 0.$$

- Both equations have the incompressibility constraint:

$$\nabla \cdot u = 0.$$

- There has recently been major progress by Elgindi [2021] on blowup for $C^{1,\alpha}$ solutions of the Euler equation and by Chen and Hou [2022] for the blowup of smooth solutions of the Euler equation on the cylinder at the boundary.
- The hypoviscous Navier–Stokes equation interpolates between Euler and Navier–Stokes. For $0 < \alpha < 1$:

$$\partial_t u + \nu(-\Delta)^\alpha u + \mathbb{P}((u \cdot \nabla)u) = 0.$$

The $\mathcal{M}$-restricted model equations

- In this talk, I will discuss the $\mathcal{M}$-restricted Euler and hypodissipative Navier–Stokes equations, where the Helmholtz projection is replaced by a projection $\tilde{\mathbb{P}}$ onto a subspace, $H^s_{\mathcal{M}}\left(\mathbb{T}^3\right) \subset H^s_{df}\left(\mathbb{T}^3\right)$:

$$\partial_t u + \tilde{\mathbb{P}}((u \cdot \nabla)u) = 0$$
$$\partial_t u + \nu(-\Delta)^\alpha u + \tilde{\mathbb{P}}((u \cdot \nabla)u) = 0.$$

- This subspace involves restricting the Fourier modes to a dyadic tree constructed iteratively using permuations.
- The amplitudes are also restricted in a way that promotes singularity formation, but the $(u \cdot \nabla)u$ nonlinearity is kept.
- This means both the energy equality and the identity for enstrophy growth are unaffected by the change in projection.

Model equations for Euler and Navier–Stokes

- Montgomery-Smith [2001] proved finite time blowup for a scalar model equation where $-\mathbb{P}\nabla \cdot (u \otimes u)$ is replaced by $(-\Delta)^{\frac{1}{2}}(u^2)$,
$$\partial_t u - \Delta u = (-\Delta)^{\frac{1}{2}}(u^2).$$
- This was generalized to blowup for a model equation on the space of divergence free vector fields by Gallagher and Paicu [2009], and further generalized to a model equation with an energy equality by Tao [2016].
- Tao's model equation is given by
$$\partial_t u - \nu \Delta u + B(u,u) = 0,$$
where
$$\langle B(u,u), u \rangle = 0$$
$$\|B(u,u)\|_{L^2} \leq \|u\|_{L^4} \|\nabla u\|_{L^4}.$$

- The dyadic Euler and dyadic Navier–Stokes equations were introduced by Katz and Pavlović [2005] and Friedlander and Pavlović [2004]. The Dyadic Navier–Stokes equation is an infinite system of ODEs for $\{\psi_n\}_{n\in\mathbb{Z}^+}$:

$$\partial_t \psi_n = -\nu\lambda^{2\alpha n}\psi_n + \lambda^n \psi_{n-1}^2 - \lambda^{n+1}\psi_n\psi_{n+1}.$$

- Note $\psi_{-1} := 0$ by convention, that $\lambda > 1$, and that the dyadic Euler equation is obtained when $\nu = 0$.
- FP proved finite-time blowup for the inviscid dyadic model.
- KP proved finite-time blowup for the inviscid model and for dyadic Navier—Stokes when $\alpha < \frac{1}{4}$.
- Cheskidov [2008] proved finite-time blowup for the dyadic model when $\alpha < \frac{1}{3}$, and global regularity when $\alpha \geq \frac{1}{2}$.

- For sufficiently smooth solutions—meaning decaying sufficiently fast as $n \to +\infty$—the dyadic Euler equation has an energy equality

$$\sum_{n=0}^{\infty} \psi_n(t)^2 = \sum_{n=0}^{\infty} \psi_n(0)^2$$

- An analogous result with dissipation holds for dyadic NS.
- We can see using telescoping series that

$$\frac{\mathrm{d}}{\mathrm{d}t} \sum_{n=0}^{\infty} \psi_n(t)^2 = -2 \lim_{n \to \infty} \lambda^{n+1} \psi_n^2 \psi_{n+1}.$$

- A sufficient condition for energy conservation is

$$\sup_{n \in \mathbb{Z}^+} \lambda^{\gamma n} |\psi_n(t)| < +\infty,$$

for some $\gamma > \frac{1}{3}$. Note that this is Onsager critical.

The dyadic model (3/3)

- If $\psi_n(0) \geq 0$, then for all $0 \leq t < T_{max}$, we have $\psi_n(t) \geq 0$
- When the ψ_n are all nonnegative, the dyadic Euler equation is structured to send energy to higher modes.
- For all $n \in \mathbb{Z}^+$, define E_n by

$$E_n = \sum_{m=n}^{\infty} \psi_m^2.$$

- It is straightforward to compute that

$$\frac{\mathrm{d}}{\mathrm{d}t} E_n(t) = 2\psi_{n-1}^2 \psi_n$$

- When ψ is nonnegative this implies that E_n is non-decreasing. The proof of blowup is based on using this fact to show that energy transfers to arbitrarily high modes in finite-time [FP 2004, KP 2005].

Permutation symmetry

- We will take the permutation of a vector to be the permutation of its entries, $P(v)_i = v_{P(i)}$. For example,

$$P_{12}(v) = (v_2, v_1, v_3).$$

  We will define the permutation of a vector field to be given by

$$v^P(x) = P(v(P^{-1}x)).$$

- Note that if $\nabla \cdot v = 0$, then $\nabla \cdot v^P = 0$.
- We will say that a vector field is permutation-symmetric if for every permutation P,

$$v^P = v.$$

- The space of permutation-symmetric vector fields is preserved by the Euler and Navier–Stokes equations as well as our restricted model equations.

The construction of the constraint Space (1/3)

Let

$$\sigma = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}.$$

For all $m \in \mathbb{Z}^+$, we will define the frequencies k^m, h^m, j^m by

$$k^m = 2^{2m}\sigma + 3^m \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}$$

$$h^m = 2^{2m+1}\sigma + 3^m \begin{pmatrix} 1 \\ 1 \\ -2 \end{pmatrix}$$

$$j^m = 2^{2m+1}\sigma + 3^m \begin{pmatrix} 2 \\ -1 \\ -1 \end{pmatrix}.$$

The construction of the constraint space (2/3)

- Note that these canonical frequencies are constructed dyadically in terms of permutations.

$$h^m = k^m + P_{12}(k^m)$$
$$j^m = k^m + P_{23}(k^m)$$
$$k^{m+1} = h^m + j^m$$

- When $n = 2m$, let

$$\mathcal{M}_n^+ = \mathcal{P}[k^m].$$

- When $n = 2m+1$, let

$$\mathcal{M}_n^+ = \mathcal{P}[h^m] \cup \mathcal{P}[j^m].$$

- Finally, let

$$\mathcal{M}^+ = \bigcup_{n=0}^{\infty} \mathcal{M}_n^+$$
$$\mathcal{M}^- = -\mathcal{M}^+$$
$$\mathcal{M} = \mathcal{M}^+ \cup \mathcal{M}^-.$$

The construction of the constraint space (3/3)

We will say that $u \in H^s_{\mathcal{M}}\left(\mathbb{T}^3;\mathbb{R}^3\right)$ if:

$$\operatorname{supp} \hat{u} \subset \mathcal{M},$$

and for all $k \in \mathcal{M}$,

$$\hat{u}(k) \in \operatorname{span}\left\{v^k\right\},$$

where

$$v^k = \frac{P_k^\perp(\sigma)}{\left|P_k^\perp(\sigma)\right|}.$$

The dyadic tree

$$\mathcal{M}_0^+ = \left\{ \begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 2 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix}, \begin{pmatrix} 0 \\ 2 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 2 \end{pmatrix} \right\}$$

$$\mathcal{M}_1^+ = \left\{ \begin{pmatrix} 3 \\ 3 \\ 0 \end{pmatrix}, \begin{pmatrix} 3 \\ 0 \\ 3 \end{pmatrix}, \begin{pmatrix} 0 \\ 3 \\ 3 \end{pmatrix}, \begin{pmatrix} 4 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 4 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 4 \end{pmatrix}, \right\}$$

$$\mathcal{M}_2^+ = \left\{ \begin{pmatrix} 7 \\ 4 \\ 1 \end{pmatrix}, \text{ and permutations} \right\}$$

$$\mathcal{M}_3^+ = \left\{ \begin{pmatrix} 11 \\ 11 \\ 2 \end{pmatrix}, \begin{pmatrix} 14 \\ 5 \\ 5 \end{pmatrix}, \text{ and permutations} \right\}$$

Dyadic properties

Proposition

For all $k \in \mathcal{M}_n^+$, $\sigma \cdot k = 3 * 2^n$. Furthermore, if $k = h + j$, for some $h, j \in \mathcal{M}^+$, then $h, j \in \mathcal{M}_{n-1}^+$. If $k = h - j$, for some $h, j \in \mathcal{M}^+$, then $h \in \mathcal{M}_{n+1}^+, j \in \mathcal{M}_n^+$

Proof.

The first statement is just a vector calculus identity. To finish the proof, it suffices to observe that for nonnegative integers $m, r, n \in \mathbb{Z}^+$: $2^m + 2^r = 2^n$ if and only if $m = r = n - 1$, and $2^n = 2^m - 2^r$ if and only if $r = n$, and $m = n + 1$. □

Local Wellposedness: the inviscid case

Theorem

For all $u^0 \in H^s_{\mathcal{M}}, s > \frac{5}{2}$, there exists a unique strong solution of the $\mathcal{M}$-restricted Euler equation, $u \in C\left([0, T_{max}); H^s_{\mathcal{M}}\right)$, where

$$T_{max} \geq \frac{1}{C_s \|u^0\|_{H^s}}.$$

This solution satisfies the energy equality

$$\|u(\cdot, t)\|_{L^2} = \left\|u^0\right\|_{L^2},$$

and the H^s upper bound

$$\|u(\cdot, t)\|_{H^s} \leq \frac{\|u^0\|_{H^s}}{1 - C_s \|u^0\|_{H^s} t}.$$

Local Wellposedness: the viscous case

Theorem

For all $u^0 \in H^s_{\mathcal{M}}, s > \frac{5}{2}$, there exists a unique strong solution of the $\mathcal{M}$-restricted NS equation, $u \in C\left([0, T_{max}); H^s_{\mathcal{M}}\right)$, where

$$T_{max} \geq \frac{1}{C_s \|u^0\|_{H^s}}.$$

This solution satisfies the energy equality

$$\frac{1}{2}\|u(\cdot,t)\|_{L^2}^2 + \nu \int_0^t \|u(\cdot,\tau)\|_{\dot{H}^\alpha}^2 \, \mathrm{d}\tau = \frac{1}{2}\|u^0\|_{L^2},$$

and the H^s upper bound

$$\|u(\cdot,t)\|_{H^s} \leq \frac{\|u^0\|_{H^s}}{1 - C_s \|u^0\|_{H^s} t}.$$

We also have higher regularity, $u \in C^\infty\left((0, T_{max}) \times \mathbb{T}^3\right)$.

Ansatz for singularity formation

- We will say that a vector field $v \in H^s_{\mathcal{M}}$ is odd, permutation symmetric, with hj-parity, if

$$u(x) = -2 \sum_{m=0}^{+\infty} \left(\psi_{2m} \sum_{k \in \mathcal{P}[k^m]} v^k \sin(2\pi k \cdot x) \right.$$

$$\left. + \psi_{2m+1} \left(\sum_{h \in \mathcal{P}[h^m]} v^h \sin(2\pi h \cdot x) + \sum_{j \in \mathcal{P}[j^m]} v^j \sin(2\pi j \cdot x) \right) \right).$$

- This class of vector fields is preserved by the dynamics of the inviscid and viscous restricted model equations.
- We will use this Ansatz to prove finite-time blowup by a reduction to the dyadic model.

Reduction to the dyadic model

$$\partial_t \psi_n = -(12\pi^2)^\alpha \mu_n^\alpha \nu \left(\sqrt{3}\right)^{2\tilde{\alpha} n} \psi_n + \sqrt{2}\pi \beta_{n-1} \left(\sqrt{3}\right)^n \psi_{n-1}^2 - \sqrt{2}\pi \beta_n \left(\sqrt{3}\right)^{n+1} \psi_n \psi_{n+1},$$

where $\nu = 0$ for the inviscid equation and

$$\tilde{\alpha} = \frac{2\log(2)}{\log(3)}\alpha$$

$$\beta_n = \frac{1}{\left(1 + \frac{1}{2}\left(\frac{3}{4}\right)^n\right)^{\frac{1}{2}}}$$

$$\mu_n = \frac{1}{\left(1 + \frac{2}{3}\left(\frac{3}{4}\right)^n\right)^{\frac{1}{2}}}$$

and by convention

$$\psi_{-1}, \beta_{-1} := 0.$$

Positivity of the Fourier coefficients

- We will say that an odd, permutation-symmetric, hj-parity vector field $u \in H^s_{\mathcal{M}}$ is coefficient positive if for all $n \in \mathbb{Z}^+$,

$$\psi_n \geq 0,$$

  and u is not identically zero.
- That is, all of the coefficients must be nonnegative, and at least one must be positive.
- It is straightforward to observe that positivity is also preserved by the dynamics of both the $\mathcal{M}$-restricted Euler and hypoviscous Navier–Stokes equations.

Finite-time blowup for the inviscid model equation

Theorem

Suppose $u^0 \in H^s_{\mathcal{M}}, s > \frac{5}{2}$ is odd, permutation-symmetric, with hj-parity, and is coefficient positive. Then the solution of the $\mathcal{M}$-restricted Euler equation blows up in finite-time, with

$$T_{max} \leq \inf_{\frac{1}{3} < r < \frac{\sqrt{2}}{\sqrt{2}+\frac{3}{2}}} \frac{\sqrt{r}}{\sqrt{2E_0}\pi\left(\sqrt{2} - \left(\sqrt{2}+\frac{3}{2}\right)r\right)} \frac{\sqrt{3r}}{\sqrt{3r}-1},$$

where

$$E_0 = \frac{1}{12}\left\|u^0\right\|_{L^2}^2.$$

Finite-time blowup for the hypoviscous model equation

Theorem

Suppose $u^0 \in H^s_{\mathcal{M}}, s > \frac{5}{2}$ is odd, permutation-symmetric, with hj-parity, and is coefficient positive, and that $\alpha < \frac{\log(3)}{6\log(2)}$. If for some $0 < \gamma < 1 - 3\tilde{\alpha}$, the Lyapunov functional satisfies

$$H_\gamma(0) := r_\gamma \sum_{n=0}^{+\infty} \left(\sqrt{3}\right)^{2\gamma n} \psi_n(0)^2 + \sum_{n=0}^{+\infty} \left(\sqrt{3}\right)^{2\gamma n} \psi_n(0)\psi_{n+1}(0)$$
$$\geq \frac{C_{\gamma,\tilde{\alpha}}^2 \nu^2 \left(r_\gamma + \frac{1}{2} + \frac{1}{2(3^\gamma)}\right)}{\left(1 - \left(\sqrt{3}\right)^{-\epsilon}\right)\pi^2},$$

then the solution blows up in finite-time

$$T_{max} \leq \frac{1}{\kappa\sqrt{H_\gamma(0)}}.$$

Theorem

Suppose $u^0 \in H^s_{\mathcal{M}}, s > \frac{5}{2}$ *is odd, permutation-symmetric, with hj-parity, and that* $\alpha \geq \frac{\log(3)}{4\log(2)}$. *Then there is a global smooth solution of the* $\mathcal{M}$*-restricted, hypodissipative Navier–Stokes equation,* $u \in C\left([0,+\infty); H^s_{\mathcal{M}}\right)$. *Furthermore, this solutions has the upper bound*

$$\|u(\cdot,t)\|^2_{\dot{H}^s} \leq \left(\frac{5}{3}\right)^s \left\|u^0\right\|^2_{\dot{H}^s} \exp\Bigg(2\sqrt{2}\pi(3^{\tilde{s}}-1)t^{\frac{1}{2}} \left(\frac{1}{\nu}\left\|\psi^0\right\|^2_{\mathcal{H}^{\tilde{\alpha}}} + \frac{C'_\alpha t}{\nu^2}\left\|\psi^0\right\|^4_{\mathcal{H}^{\tilde{\alpha}}} \exp\left(\frac{2C_\alpha}{\nu^2}\left\|u^0\right\|^2_{L^2}\right)\right)^{\frac{1}{2}}\Bigg),$$

where $\tilde{s} = \frac{2\log(2)}{\log(3)} s$.

Relationship to the dyadic model

- Note that we have proven finite-time blowup when $\tilde{\alpha} < \frac{1}{3}$ and global regularity when $\tilde{\alpha} \geq \frac{1}{2}$.
- This corresponds exactly to the results of Cheskidov [2008] for dyadic Navier–Stokes.
- Once the reduction to the dyadic model is accomplished with our blowup Anstaz, we can essentially follow Cheskidov's proof with a some minor technical variations involving the correction factors β_n, μ_n.

Theorem

Suppose $u \in C\left([0, T_{max}); H^s_{\mathcal{M}}\right), s > \frac{5}{2}$ *is an odd permutation symmetric, hj-parity solution of the* $\mathcal{M}$*-restricted Euler or hypodissipative Navier–Stokes equation. Then for all* $0 \leq t < T_{max}$

$$\|u(\cdot, t)\|^2_{\dot{H}^s} \leq \left(\frac{5}{3}\right)^s \|u^0\|^2_{\dot{H}^s} \exp\left(2\sqrt{2}\pi(3^{\tilde{s}} - 1) \int_0^t \sup_{n \in \mathbb{Z}^+} \left(\sqrt{3}\right)^n \psi_n(\tau)\, \mathrm{d}\tau\right),$$

where $\tilde{s} = \frac{2\log(2)}{\log(3)} s$*. In particular, if* $T_{max} < +\infty$*, then*

$$\int_0^{T_{max}} \sup_{n \in \mathbb{Z}^+} \left(\sqrt{3}\right)^n \psi_n(t)\, \mathrm{d}t = +\infty.$$

Singularity at the origin

Theorem

Suppose $u \in C\left([0,T_{max}); \dot{H}^s_{\mathcal{M}}\right), s > \frac{5}{2}$ *is an odd permutation symmetric, hj-parity, coefficient positive solution of the* $\mathcal{M}$*-restricted Euler or hypodissipative Navier–Stokes equation. Then for all* $0 \leq t < T_{max}$*,*

$$\nabla u(\vec{0},t) = \lambda(t) \begin{pmatrix} 0 & -1 & -1 \\ -1 & 0 & -1 \\ -1 & -1 & 0 \end{pmatrix}$$

$$\lambda(t) = 12\sqrt{2}\pi \sum_{n=0}^{+\infty} \psi_n(t) \frac{\left(\sqrt{3}\right)^n}{\left(1+\frac{2}{3}\left(\frac{3}{4}\right)^n\right)^{\frac{1}{2}}} \geq 0.$$

Furthermore, if $T_{max} < +\infty$*, then*

$$\int_0^{T_{max}} \lambda(t)\,\mathrm{d}t = +\infty.$$

Proof.

We will see from geometric considerations involving permutation symmetry that

$$\nabla u(\vec{0},t) = \lambda(t) \begin{pmatrix} 0 & -1 & -1 \\ -1 & 0 & -1 \\ -1 & -1 & 0 \end{pmatrix}$$

$$\lambda(t) = 12\sqrt{2}\pi \sum_{n=0}^{+\infty} \psi_n(t) \frac{\left(\sqrt{3}\right)^n}{\left(1+\frac{2}{3}\left(\frac{3}{4}\right)^n\right)^{\frac{1}{2}}} \geq 0.$$

We know that $\psi_n(t) \geq 0$, so we can see that

$$\lambda(t) = 12\sqrt{2}\pi \sum_{n=0}^{+\infty} \psi_n(t) \frac{\left(\sqrt{3}\right)^n}{\left(1+\frac{2}{3}\left(\frac{3}{4}\right)^n\right)^{\frac{1}{2}}} \geq C \sup_{n\in\mathbb{Z}^+} \left(\sqrt{3}\right)^n \psi_n(t)$$

□

Singularity at the origin: discussion

- Note that the gradient at the origin is symmetric, with $\omega(\vec{0},t)=0$ and

$$S(\vec{0},t)=\lambda(t)\begin{pmatrix} 0 & -1 & -1 \\ -1 & 0 & -1 \\ -1 & -1 & 0 \end{pmatrix}$$

- The strain matrix at the origin has eigenvalues $-2\lambda, \lambda, \lambda$, with axial compression along $\operatorname{span}(\sigma)$, and planar stretching in the plane orthogonal to σ.
- This is exactly the structure of singularity formation predicted by the regularity criteria in terms of λ_2^+, the positive part of the middle eigenvalue [Neustupa-Penel 2001].

Proposition

Suppose $u \in H^s_{df}, s > \frac{5}{2}$ *is permutation-symmetric. Then for all* $1 \leq i, j \leq 3, i \neq j$,

$$\partial_i u_j(\vec{0}) = \partial_1 u_2(\vec{0}),$$

and for all $1 \leq i \leq 3$,

$$\partial_i u_i(\vec{0}) = 0.$$

Therefore

$$\nabla u(\vec{0}) = \partial_1 u_2(\vec{0}) \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix}.$$

- Note that this implies that $\lambda(t) = -\partial_1 u_2(\vec{0})$, in the preceding theorem.

Fourier mode computation

Proposition

Fix $k \in \mathbb{Z}^3, v \in \mathbb{R}^3$, *such that* $k \cdot v = 0$. *Let*

$$u(x) = -\sum_{P \in \mathcal{P}_3} P(v) \sin\left(2\pi P(k) \cdot x\right).$$

Then u *is permutation-symmetric and divergence free, and*

$$-\partial_1 u_2(\vec{0}) = 2\pi(\sigma \cdot k)(\sigma \cdot v)$$

Proof.

$$\begin{aligned} \partial_1 u_2(\vec{0}) &= -2\pi \sum_{i \neq j} k_i v_j - 2\pi \sum_i k_i v_i \\ &= -2\pi \sum_{i,j} k_i v_j = -2\pi(\sigma \cdot k)(\sigma \cdot v) \end{aligned}$$

Open question

Question

Fix initial data

$$u^0 = \begin{pmatrix} 1 \\ -2 \\ -5 \end{pmatrix} \sin\left(2\pi \begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix} \cdot x\right) + \text{all permutations.}$$

The $\mathcal{M}$-restricted Euler equation blows up with initial data u^0, as does the $\mathcal{M}$-restricted hypodissipative Navier–Stokes equation for all $\alpha < \frac{\log(3)}{6\log(2)}$, for sufficiently small viscosity, $\nu < \nu_\alpha$.

Do the actual Euler or hypodissipative Navier–Stokes equations exhibit finite-time blowup with this initial data?

Thank you

Thanks to the organizers for the chance to be back at MSRI in person and for the chance to speak.