Limiting Configurations for Solutions to the 1D Euler Alignment System

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Small-scale interactions $\overset{How?}{\Longrightarrow}$ Large-scale structures

Figure: A flock of starlings. (Public domain photo, from Wikipedia)

The Cucker–Smale and Euler Alignment Systems

Cucker–Smale system (CS):

$$\begin{cases} \dot{x}_i = v_i \\ m_i \dot{v}_i = \displaystyle\sum_{j=1}^{N} m_i m_j \phi(x_i - x_j)(v_j - v_i) \end{cases}$$

- $(m_i, x_i, v_i) = (\text{mass}, \text{position}, \text{velocity})$ of ith *agent*
- $\phi =$ *communication protocol*, assumed nonnegative, radially decreasing.

Euler Alignment system (EA): a hydrodynamic version of (CS).

$$\begin{cases} \partial_t \rho + \nabla \cdot (\rho \mathbf{u}) = 0 \\ \partial_t (\rho \mathbf{u}) + \nabla \cdot (\rho \mathbf{u} \otimes \mathbf{u}) = \displaystyle\int_{\mathbb{R}^d} \rho(x)\rho(y)\phi(x-y)(\mathbf{u}(y) - \mathbf{u}(x))\, dy. \end{cases}$$

- $\rho =$ density, $\mathbf{u} =$velocity.

$\phi \in L^1_{\mathrm{loc}}(\mathbb{R}) \rightsquigarrow$ (EA) has 'hyperbolic character'

$\phi \notin L^1_{\mathrm{loc}}(\mathbb{R}) \rightsquigarrow$ (EA) has 'parabolic character'

Flocking

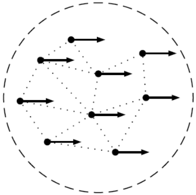
(CS) and (EA) generate interest primarily because of their long-time behavior.

We say that **(Strong) Flocking** occurs if, as $t \to +\infty$, we have

– discrete version: $x_i(t) - x_j(t) \to \overline{x_{ij}}, \quad \forall i, j,$

– continuum version: $\rho(x - \overline{u}t, t) \to \overline{\rho}(x).$

(can take $\overline{u} = 0$ by Galilean invariance)



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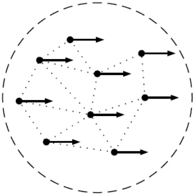


Strong Flocking occurs (at least weak-$*$) if ϕ is **heavy-tailed**:

$$\int_0^\infty \phi(r)\,\mathrm{d}r = +\infty \qquad \text{(heavy-tail condition).}$$

(This is a 'sledgehammer' assumption, but it is often useful.)

Figure courtesy of Roman Shvydkoy.

Main Goal

We want to understand what the density profile looks like after a long time. We are particularly interested in the formation of **high density** regions, or more specifically, **clustering** of mass at a single point.

- When $\phi \in L^1_{\text{loc}}$, **Dirac masses can appear in finite time**, even for smooth data.
- **Time-asymptotic 'mass concentration'** can occur even if solutions remain smooth for all finite time.

We want to predict where these occur **from the initial conditions**.

We restrict attention to the **1D case**, by technical necessity.

Outline

- Background: Smooth solutions
  - Critical Threshold Conditions
  - Time-asymptotic concentration of mass
- Weak Solutions
  - Atomic Solutions to (EA) and the sticky particle collision rules
  - Scalar balance law formulation
  - Existence and Uniqueness: skeleton of the proof
- Mass clustering
  - Generalized inverse as a ‘flow map’; definition of a cluster
  - The subcritical, supercritical, and critical regions
  - The discrete level
  - Proof of clustering in the supercritical region

Background

Critical Threshold Condition (CTC)

Define

$$e = \partial_x u + \phi * \rho.$$

Then

$$\partial_t e + \partial_x (ue) = 0.$$

The (CTC) says (Carrillo-Choi-Tadmor-Tan (2016); Tan (2019))

- $e^0 > 0$ everywhere $\rightsquigarrow$ global-in-time classical solution to (EA)
- $e^0 < 0$ somewhere $\rightsquigarrow$ finite-time blowup
- $e^0 = 0$

  $\rightsquigarrow$ no finite-time blowup, if ϕ is bounded

  $\rightsquigarrow$ finite-time blowup is possible, if ϕ is *weakly singular*:

$$\phi(r) \sim cr^{-\beta}, \quad r \in (0, R), \ \beta \in (0, 1) \qquad \text{(weakly singular } \phi\text{).}$$

Comparing Characteristics; Pressureless Euler and (EA)

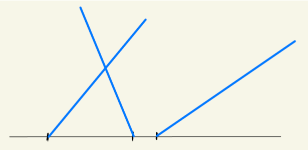
The antiderivative of e gives a slightly different perspective (L 2020) :

$$\psi := u + \Phi * \rho, \qquad \Phi(x) = \int_0^x \phi(y)\,dy.$$

1D Pressureless Euler (PE)

$$\partial_t \rho + \partial_x(\rho u) = 0$$
$$\partial_t(\rho u) + \partial_x(\rho u^2) = 0$$

- Characteristics $x(\cdot, t)$ transport u.
- Crossing $\iff$ u^0 decreases



1D Euler Alignment (EA)

$$\partial_t \rho + \partial_x(\rho u) = 0$$
$$\partial_t(\rho\psi) + \partial_x(\rho u \psi) = 0$$

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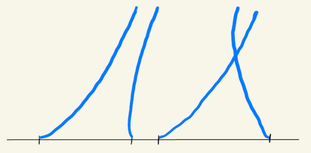

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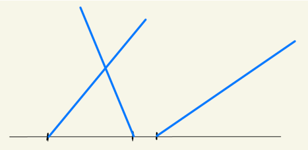

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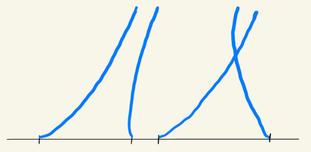


If ϕ is bounded and heavy-tailed, and ψ^0 is nondecreasing, then (Lear et. al 2022):

$$\boxed{\overline{x}(\beta) - \overline{x}(\alpha) \sim \psi^0(\beta) - \psi^0(\alpha),} \qquad \overline{x} := \lim_{t\to\infty} x(\cdot, t)$$

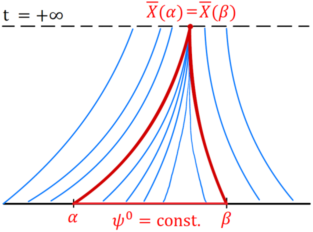
Concentration of Mass

The relation

$$\overline{x}(\beta) - \overline{x}(\alpha) \sim \psi^0(\beta) - \psi^0(\alpha)$$

allows us to predict and manipulate the flocking state $\bar{\rho}$ via the initial data.

Special interest: If $\psi^0(\alpha) = \psi^0(\beta)$ gives rise to **concentration of mass**.



Theorem (Lear–L–Shvydkoy–Tadmor (*Adv. Math.* 2022))

Suppose ϕ is bounded and heavy-tailed; denote $\mathcal{Z} = \{\partial_x \psi^0 = 0\}$. Then

$$\rho(t) \overset{*}{\rightharpoonup} \bar{\rho} = f\,dx + \mu \qquad \text{as } t \to \infty,$$

where $\mu \perp dx$ and

$$\mu = \overline{x}_\sharp(\rho^0 \chi_{\mathcal{Z}}\, dx).$$

Applications: Unidirectional solutions, 'mass concentration sets' of fractional dimension...

Weak Solutions

Life after blowup? Nonuniqueness of Distributional Weak Solutions

Trajectories can collide in finite time $\rightsquigarrow$ look for measure-valued $\rho(t)$.

The simplest solutions with measure-valued density are atomic:

$$\rho_N(x,t) = \sum_{i=1}^{N} m_{i,N}\delta(x - x_{i,N}(t)), \qquad \rho_N u_N(x,t) = \sum_{i=1}^{N} m_{i,N} v_{i,N}(t)\delta(x - x_{i,N}(t)).$$

If $(x_{i,N}(t), v_{i,N}(t))_{i=1}^{N}$ obey Cucker–Smale, then (ρ_N, u_N) is a weak solution.

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These are *not* the only atomic solutions! We may impose **any collision rule that conserves momentum** without violating the weak formulation.

- ‘Free-flow’ dynamics (particles don’t notice collisions)
- Elastic collisions
- Inelastic (‘sticky particle’) collisions
- Something in between?

We need a *selection principle* to get uniqueness.

Life after blowup? Selecting a Unique Weak Solution

Sticky particle (i.e., completely inelastic) collision rules dissipate the most kinetic energy and play a role in the pressureless Euler theory.

We seek a selection principle that is

- Consistent with sticky particle Cucker–Smale (SPCS), and
- (More importantly) compatible with the limit $N \to \infty$.

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Once we have solutions, we want to understand where Dirac masses form.

We look to pressureless Euler system ($\phi \equiv 0$) for inspiration. In this case, the 'free flow' dynamics are straight line paths.

We recall that the pressureless Euler system reads:

$$\partial_t \rho + \partial_x(\rho u) = 0$$
$$\partial_t(\rho u) + \partial_x(\rho u^2) = 0.$$

(Partial) Literature on 1D Pressureless Euler

1994, Grenier: Existence via subsequential limit of atomic solutions.

1996, E–Rykov–Sinai: “Generalized Variational Principle” (GVP)

1998, Brenier–Grenier: Reduction to a scalar conservation law, Uniqueness*

1999, Bouchut–James: ‘Duality solutions’

2001, Huang–Wang: Upgraded GVP, explicit(-ish) formula for unique(!) solution. (Main) uniqueness criterion is a one-sided Lipschitz condition on u.

2009, Natile–Savaré: Solution obtained explicitly through an L^2 projection of the ‘free-flow’ dynamics onto the cone of monotone maps.

2015, Cavalletti–Sedjro–Westdickenberg: Natile–Savaré simplified

More works:

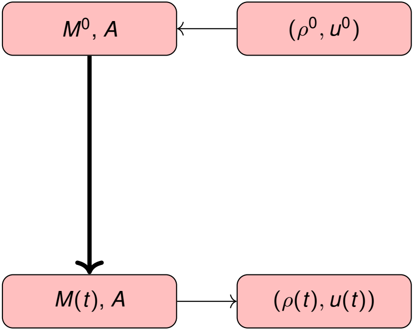
– Probabilistic viewpoint: Dermoune 1999, Moutsinga 2008, Hynd 2019/20/22
– Extensions to 1D Euler–Poisson: Nguyen–Tudorascu 2008/15, Brenier–Gangbo–Savaré–Westdickenberg 2012
– Extensions to 1D (EA) with $\phi \equiv 1$ (i.e., pressureless Euler with local damping): Ha–Huang–Wang 2014, Jin 2015/16

1D Weak Solutions to (EA), à la Brenier–Grenier

Theorem (L–Tan (*Comm. PDE's* 2023))

Given $\rho^0 \in \mathcal{P}_c(\mathbb{R})$, $u^0 \in L^\infty(d\rho^0)$,

- *There exists a unique 'entropy solution'* (ρ, u) *of (EA).*



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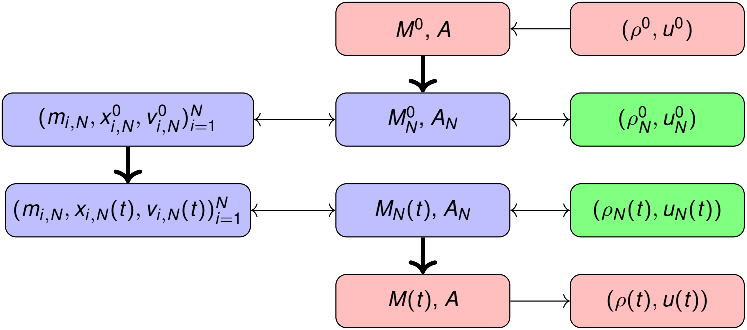
Given $\rho^0 \in \mathcal{P}_c(\mathbb{R})$, $u^0 \in L^\infty(d\rho^0)$,

- *There exists a unique 'entropy solution' (ρ, u) of (EA).*
- *This solution can be approximated by sticky particle Cucker–Smale atomic solutions (ρ_N, u_N) generated from atomic initial data. The discrete solutions are themselves entropy solutions.*



The Scalar Balance Law: Derivation and Uniqueness

Derivation of the *M*-equation:

$$\begin{cases} \partial_t \rho + \partial_x(\rho u) = 0 \\ \partial_t(\rho\psi) + \partial_x(\rho\psi u) = 0 \end{cases} \Longrightarrow \begin{cases} \partial_t M + u\partial_x M = 0 \\ \partial_t Q + u\partial_x Q = 0 \end{cases}$$

$$(\psi = u + \Phi * \rho) \qquad (\partial_x M = \rho,\ \partial_x Q = \rho\psi).$$

[1] There is very little meaningful 'choice' involved here.

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Choose[1] A so that $Q^0 = A(M^0)$. Then $Q \equiv A(M)$ if everything is smooth, and

$$\partial_t M + \partial_x(A(M)) = -\rho u + \rho\psi = \rho\Phi * \rho = (\Phi * \partial_x M)\partial_x M.$$

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Entropy condition: For η convex and q such that $\eta' A' = q'$,

$$\partial_t \eta(M) + \partial_x q(M) \leq (\Phi * \partial_x M)\partial_x \eta(M) \qquad \text{(distributional sense)}$$

Given suitable (M^0, A)**, any entropy solution** M **must be unique.** (Proof: Kruzkov argument, with significant new difficulties for the nonlocal RHS.)

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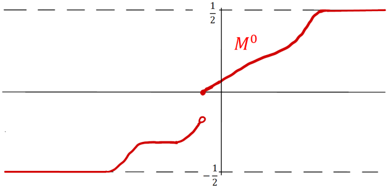
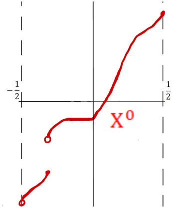
The Scalar Balance Law: Existence—Data and Flux

We take M^0 to be a (shifted) CDF of ρ^0 and X^0 its generalized inverse.





$$M^0(x) = -\tfrac{1}{2} + \rho^0((-\infty, x]),$$

$$X^0(m) = \inf\{x \in \mathbb{R} : M^0(x) \geq m\},$$

$$m \in (-\tfrac{1}{2}, \tfrac{1}{2}] = \text{ set of 'mass labels.'}$$

More generally, we define

$$X(m, t) = \inf\{x \in \mathbb{R} : M(x, t) \geq m\}.$$

(This is a 'weak flow map.')

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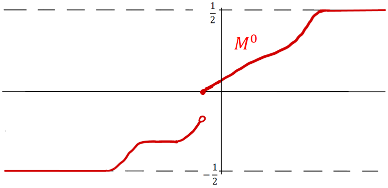

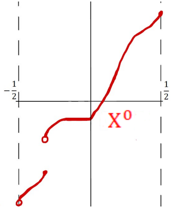

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More generally, we define

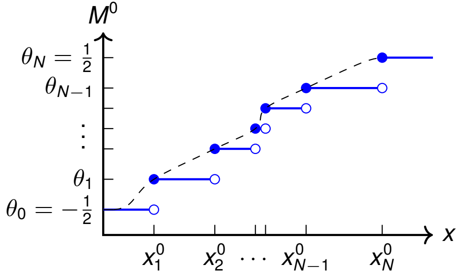
$$X(m,t) = \inf\{x \in \mathbb{R} : M(x,t) \geq m\}.$$

(This is a 'weak flow map.') We define the flux A by

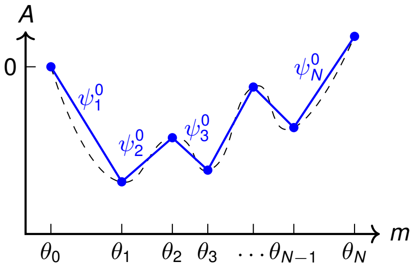
$$A(m) = \int_{-\frac{1}{2}}^{m} \psi^0 \circ X^0(\widetilde{m})\, \mathrm{d}\widetilde{m}.$$

Note that $A(M^0(x)) = \rho^0\psi^0((-\infty, x]) =: Q^0(x)$.

The Scalar Balance Law: Existence—Sticky Particle Discretization



M^0 (dashed) and M_N^0 (solid).



A (dashed) and A_N (solid).

(Initial) discretization process:

(1) Pick $(m_i)_{i=1}^N$; set $\theta_i = \sum_{j=1}^i m_j$.

(2) Put $x_i^0 = \inf\{x : M^0(x) \geq \theta_i\}$.

(3) Set $M_N^0(x) = \sum_{j=1}^i \theta_j 1_{[x_{j-1}^0, x_j^0)}(x)$.

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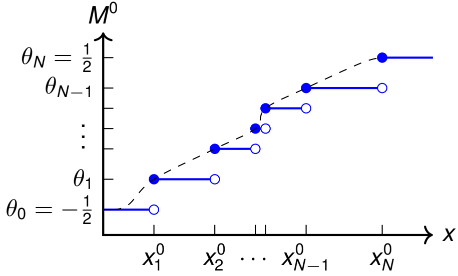

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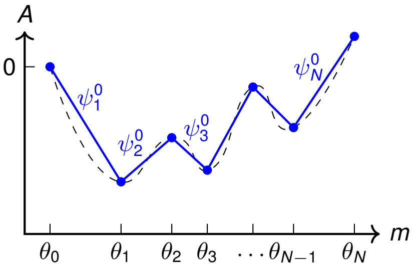

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(3) Set $M_N^0(x) = \sum_{j=1}^i \theta_j 1_{[x_{j-1}^0, x_j^0)}(x)$.

(4) Put $A_N(\theta_i) = A(\theta_i)$, interpolate.

(5) $\psi_i^0 = \frac{A(\theta_i) - A(\theta_{i-1})}{\theta_i - \theta_{i-1}}$.

(6) $v_i^0 = \psi_i^0 - \sum_{j=1}^N m_j \Phi(x_i^0 - x_j^0)$.

This gives us initial data $(m_i, x_i^0, v_i^0)_{i=1}^N$ for sticky particle Cucker–Smale, and discretized data/flux (M_N^0, A_N) for the scalar balance law.

Sticky Particle Discretization

Let $(m_i, x_i(t), v_i(t))$ be the sticky particle (CS) solution; define

$$J_i(t) = \{j : x_i(t) = x_j(t)\}, \qquad \text{(agents stuck to agent } i\text{)},$$

$$\psi_i(t) = v_i(t) + \sum_{j=1}^{N} m_j \Phi(x_i(t) - x_j(t)).$$

Completely inelastic collision rules imply

$$v_i(t) = \frac{\sum_{j \in J_i(t)}^{N} m_j v_j(t-)}{\sum_{j \in J_i(t)}^{N} m_j} \implies \psi_i(t) = \frac{\sum_{j \in J_i(t)}^{N} m_j \psi_j(s)}{\sum_{j \in J_i(t)}^{N} m_j}, \qquad 0 \le s \le t.$$

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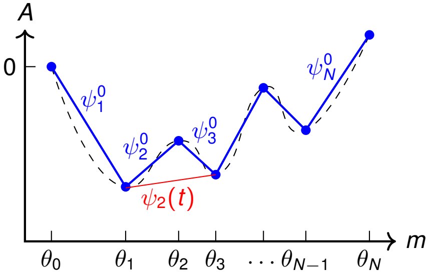

Pictured: $J_2(t) = J_3(t) = \{2, 3\}$.

Sticky Particle Discretization

Discretized scalar balance law solution:

$$M_N(x,t) = \sum_{j=1}^{i} \theta_j \mathbf{1}_{[x_{j-1}(t), x_j(t))}(x)$$

Atomic solution of (EA):

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Convergence:

$$M_N(t) - M(t) \to 0 \text{ in } L^1(\mathbb{R}), \qquad \partial_t M_N \overset{*}{\rightharpoonup} \partial_t M \text{ in } \mathcal{M}(\mathbb{R}),$$

$$\Longrightarrow \mathcal{W}_1(\rho_N(t), \rho(t)) \to 0, \qquad \rho_N u_N \overset{*}{\rightharpoonup} \rho u \text{ in } \mathcal{M}(\mathbb{R}).$$

Mass Clustering

Clustering Definitions

The generalized inverse of $M(t)$ encodes the 'location' of each mass label m:

$$X(m,t) = \inf\{x : M(x,t) \geq m\}.$$

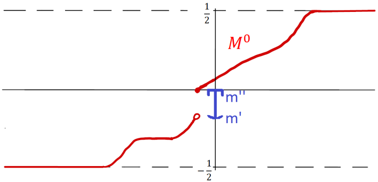
Since $X(\cdot,t)$ is left-continuous, we define clusters as half-open intervals.

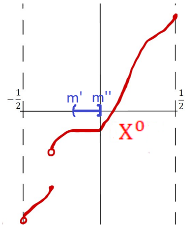
Definition (Finite- and Infinite-Time Clusters)

– Suppose $X(\cdot,t)$ is constant on a (largest) interval $(m', m'']$ containing m. Then $(m', m'']$ is the ***t*-cluster at** m. An **initial cluster** is a 0-cluster.

– Suppose $\operatorname{diam} X(I,t) \xrightarrow{t\to\infty} 0$ for some (largest) interval $I = (m', m'']$ or $I = (m', m'')$ containing m. Then I is the **infinite-time cluster at** m.

Example: The interval $(m', m'']$ is an initial cluster.





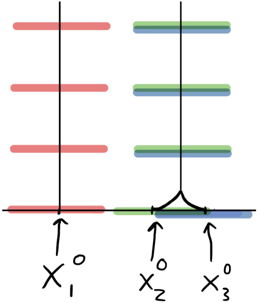
Q: Where do clusters form?

No Free Lunch: Projecting Free-Flow Dynamics Doesn't Work

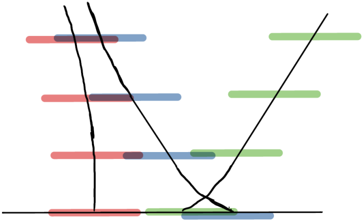
Projecting free-flow Cucker–Smale dynamics onto the cone of monotone maps does *not* yield sticky particle Cucker–Smale.

$\implies$ Approach of Natile–Savaré does *not* yield the entropy solution of (EA), so we can't find the clusters this way. (C.f. Ha et. al 2019.)

Sticky particle Cucker–Smale

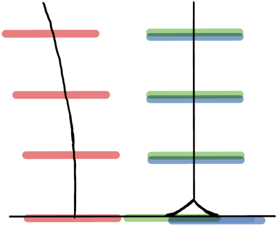


Cucker–Smale



$\rightsquigarrow$

Projected Cucker–Smale



Building off the CTC

Q (again): Where do mass clusters form?

Formally,

$$A''(m) = \frac{\mathrm{d}}{\mathrm{d}m}\psi^0(X^0(m)) = e^0(X^0(m)) \cdot (X^0)'(m).$$

So $e^0 \geq 0$ (everywhere) " $\iff$ " A is convex.

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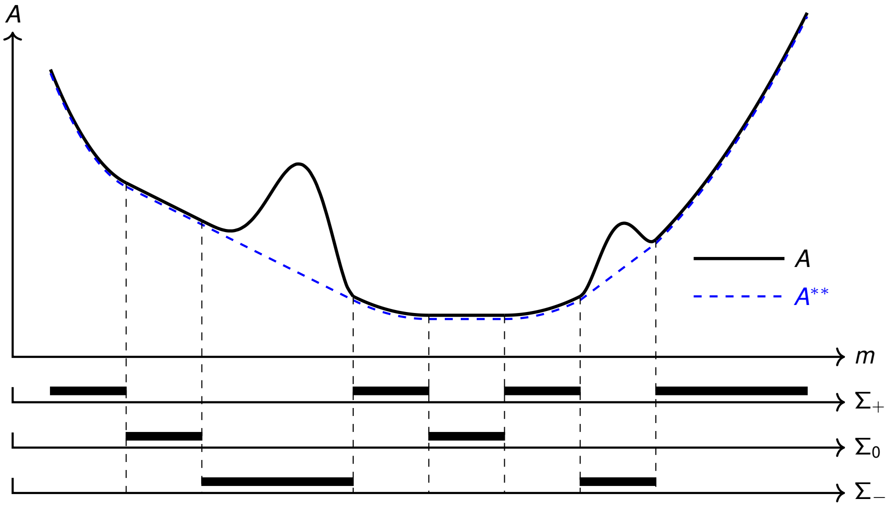
Prediction: The answer to our question should involve (non)convexity of A.

Notation: The **lower convex envelope** of A will be denoted by A^{**}.

Three Regimes: An Illustration



Three Regimes and Their Clustering Behaviors

Q (again): Where do mass clusters form?

We can answer this question (though not at a specific t) by looking at the **lower convex envelope** A^{**} of A, and dividing the interval $(-\frac{1}{2}, \frac{1}{2}]$ as follows.

Definition

- *Subcritical region* ($\approx \{(A^{**})'' > 0\}$):

  $$\Sigma_+ = \{m \in (-\tfrac{1}{2}, \tfrac{1}{2}] : A^{**} \text{ is not linear on any interval of the form } (m', m]\}$$

- *Critical region*: $$\Sigma_0 = \bigcup_{\substack{A \text{ is linear and} \\ \text{equal to } A^{**} \text{ on } (m', m'']}} (m', m''].$$

- *Supercritical region*: $$\Sigma_- = \{m \in (\tfrac{1}{2}, \tfrac{1}{2}) : A(m) > A^{**}(m)\}.$$

Simplified statement of results:

- Clustering never occurs on Σ_+.
- Finite-time clustering occurs at every point of Σ_-.
- Other clustering behavior depends on ϕ.

Statement of Results

Theorem (L–Tan)

Assume $\phi \in L^1_{\mathrm{loc}}$ and A is convex in a neighborhood of any $m \in \partial\Sigma_-$. Then

I. *There is no finite- or infinite-time clustering at any point of Σ_+.*

II. *If m and $\widetilde{m}$ lie in the same connected component of Σ_-, then there exists $T > 0$ such that $X(m,t) = X(\widetilde{m},t)\ \forall t \geq T$.*

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   (iii) *Heavy-tailed, weakly singular ϕ, $m \notin \Sigma_+$. There exists $T > 0$ such that $L(m)$ is a t-cluster $\forall t \geq T$.*

The Discrete Level

Recall $J_i(t)$ encodes the indices of agents 'stuck' to agent i. Denote

$$i_*(t) = \min J_i(t), \qquad i^*(t) = \max J_i(t).$$

We choose breakpoints $\{\theta_{i,N}\}_{i=0}^{N}$ so that

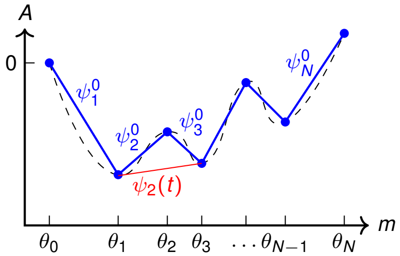
$$\{A_N > A_N^{**}\} = \{A > A^{**}\} = \Sigma_-.$$

Lemma

Suppose $\theta_i \notin \Sigma_-$ and $i^*(t) > i$ for some smallest time $t \in [0, \infty)$. Then A^{**} is linear on $[\theta_{i_*(t)-1}, \theta_{i^*(t)}]$, and the endpoints of this interval do not lie in Σ_-.

Corollary

If $J_i(t) \neq \{i\}$, then A^{**} is linear on $[\theta_{i_*(t)-1}, \theta_{i^*(t)}]$.



The Lemma and Corollary guarantee that agents 1 and 2 do not collide in finite time.

The Corollary shows that, at least at the discrete level, collisions are confined to a single $L(m)$.

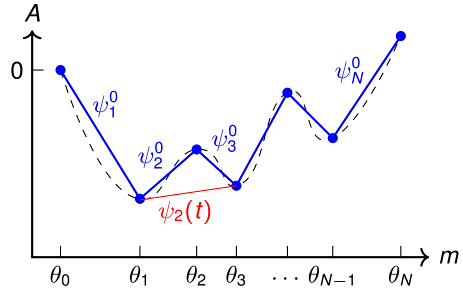
The Discrete Level, cont.

Proposition

Suppose (θ_i, θ_j) is a connected component of Σ_-. Then agents $i+1$ and j collide in some finite time.

Proof:

$$\frac{\mathrm{d}}{\mathrm{d}t}(x_j - x_{i+1}) = \underbrace{\psi_j - \psi_{i+1}}_{\substack{\text{show this is}\\ \leq -c<0\\ \text{before collision}\\ \text{of } i+1 \text{ and } j}} \underbrace{- \sum_{\ell=1}^{N} m_\ell \int_{x_{i+1}}^{x_j} \phi(y - x_\ell)\,\mathrm{d}y}_{\geq 0}$$



Convergence of Discretization

Define

$$X_N(m,t) = \inf\{x : M_N(x,t) \geq m\}. \qquad (\text{so } X_N(\theta_i,t) = x_i(t).)$$

We have

$$\|X_N(t) - X(t)\|_{L^1(\mathbb{R})} = \|M_N(t) - M(t)\|_{L^1(\mathbb{R})} \to 0, \qquad \text{as } N \to \infty,$$

so

$$X_{N_k}(m,t) \to X(m,t) \text{ a.e.}, \qquad \text{as } k \to \infty.$$

We leverage this a.e. convergence and the monotonicity of $X(\cdot, t)$ to upgrade our conclusions from the discrete to continuous level.

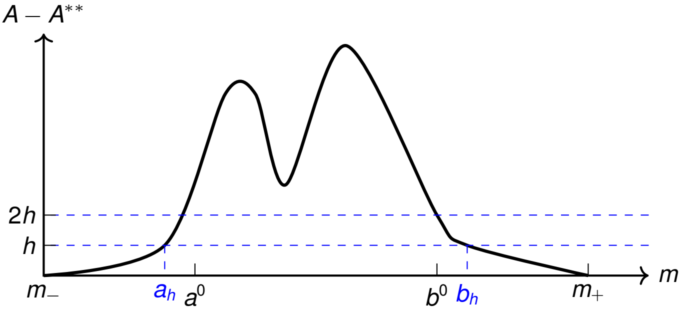
There are, however, additional subtleties involved in the continuous level (especially the ϕ-dependent statement III).

Proof of Main Theorem, part II. (Sketch)

Let (m_-, m_+) be a connected component of Σ_-.

Difficulty: m_- and m_+ might never belong to same cluster. $\implies$ Move inward.

- For small h, we have $(A - A^{**})^{-1}(h) \cap [m_-, m_+] = \{a_h, b_h\}$ and $a_h < a^0 < b^0 < b_h$, where A is convex on $[m_-, a^0]$, $[b^0, m_+]$.
- For fixed h, we want to show $X(b_h, t) - X(a_h, t) \to 0$ in finite time.



Proof of Main Theorem, part II. (Sketch)

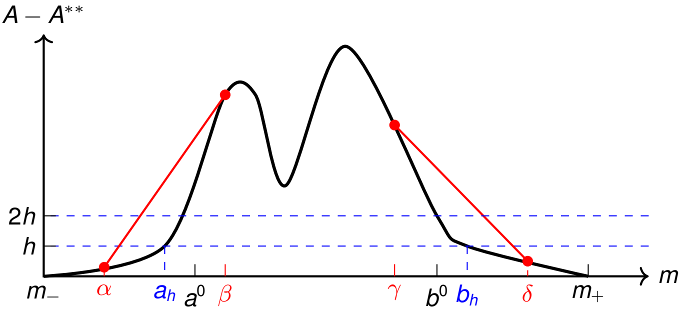
'Morally,' if $(\alpha, \beta]$ and $(\gamma, \delta]$ are clusters at a_h and b_h, their difference in ψ (think 'velocity') is the difference in the slopes of the red segments.

$\alpha, \beta, \gamma, \delta$ depend on time (clusters will grow), but we can get a lower bound on the difference of these slopes as long as

$$\boxed{m_- \leq \alpha} < a_h \leq \beta \leq \gamma < \beta_h \leq \boxed{\delta \leq m_+.}$$

This will give us an upper bound on the collision time between the clusters.

The boxed inequalities and a rigorous version of the above idea are available when we discretize.



Thanks for your attention!

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