How to Incentivize Hospitals in Dynamic Kidney Exchange

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1. Dynamically efficient mechanism of a single hospital (Ünver (2010) has already purposed) **full decentralization**
2. Multi-hospital Dynamic Kidney Exchange

Outline

1. Dynamically efficient mechanism of a single hospital (Ünver (2010) has already purposed) **full decentralization**
2. Multi-hospital dynamic kidney exchange:
   - The combination of two hospitals **centralization**
   - A multi-hospital dynamic exchange mechanism: Threshold with Inter-hospital-Exchange(TIE) **partially centralization**

Research Questions and Motivation

- How should we design a mechanism to increase patients' welfare through mitigating the waiting time for kidney transplants?
- Do hospitals always match more number of patient-donor pairs through pooling with the other hospital centralization than matching their own pairs individually full decentralization? **No**
- When does a hospital prefer matching its arriving pairs individually over matching them through pooling with the other hospital?
- Is there an incentive compatible mechanism of inter-hospital exchange such that hospitals prefer participating these inter-hospital exchanges over full decentralization, that is matching their pairs separately? **Threshold Inter-hospital Exchange Mechanism**

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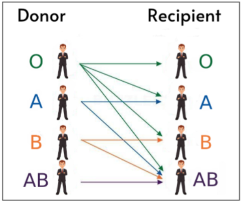
Model

- There are two compatibility requirements for a transplantation:
  - ABO blood-type: An O blood type patient can only receive a kidney from an O donor,
  - an A blood type patient can receive a kidney from an O type donor or an A type donor,
  - a B blood type patient can only receive a kidney from an O or a B blood type donor, and
  - an AB blood type patient can receive a kidney from all blood type donors.
  - Tissue type: The presence of antibodies, called a positive crossmatch, effectively rules out transplantation. A patient may not receive a kidney from its donor because of tissue-type incompatibility even though blood types are compatible. A blood type compatible pair can participate in an exchange if and only if the pair is tissue-type incompatible.



Incompatible Pairs and Assumptions

- The type of a pair is denoted by $X - Y$, where X is the blood type of patient and Y is the blood type of its donor.
- **Matching, exchange:** A matching is a set of exchanges. A patient receives a compatible donor, each pair is a part of at most one exchange. A matching with the maximum number of pairs is **maximal**.
- Let $\mathcal{T}$ be the set of all pair types.
- We partition $\mathcal{T}$ into the sets of ***overdemanded***, ***underdemanded***, **self-demanded** and ***reciprocally demanded*** pair types:

$$P^O = \{A - O, B - O, AB - O, AB - A, AB - B\}$$
$$P^U = \{O - A, O - B, O - AB, A - AB, B - AB\}$$
$$P^S = \{A - A, B - B, O - O, AB - AB\}$$
$$P^R = \{A - B, B - A\}.$$

- In Table 1, these probabilities obtained from simulations based on the data from the U.S. Organ Procurement and Transplant Network (OPTN) and the Scientific Registry of Transplant Recipients (SRTR) 2003 Annual Report, covering the period 1993-2002 (`http://www.optn.orgon11/22/2004`).

Patient Blood Type	Donor Blood type	Frequency
O	O	11.36
O	A	30.71
O	B	13.09
O	AB	3.50
A	O	7.57
A	A	5.46
A	B	9.33
A	AB	2.43
B	O	3.34
B	A	9.28
B	B	1.03
B	AB	1.04
AB	O	0.92
AB	A	0.64
AB	B	0.25
AB	AB	0.066

Table 1

Assumptions

Assumption

(Limit Assumption)
No patient is tissue-type incompatible with the donor of another pair.

Assumption

(Long Run assumption) Under any dynamic matching mechanisms in the long run, there is an arbitrarily large number of underdemanded pairs from each pair type in the exchange.

Assumption

(Generic assumption) The probabilities p_{A-B} and p_{B-A} are sufficiently close to each other.

Assumption

(No self-demanded pairs) There are no self-demanded pair types available for exchange.

Exchange Arrival Probabilities Assumption

Assumption

For each pair type $X - Y \in \mathcal{T}$, we denote the pair's arrival probability by p_{X-Y}, and the associated probability distribution by $p = (p_{X-Y})_{X-Y\in\mathcal{T}}$.

- *(Arrival probabilities of pair types)*
  - *(i) The arrival probability of an $A - B$ pair is higher than the arrival probability of a $B - A$ pair: $p_{A-B} > p_{B-A}$.*
  - *(ii) The overdemanded pairs that can be matched with an $A - B$ pair in larger exchanges arrive less frequently than those that can be matched with a $B - A$ pair. That is,*

$$\sum_{X-Y\in P^{O}(A-B)} p_{X-Y} < \sum_{X-Y\in P^{O}(B-A)} p_{X-Y}. \qquad (1)$$

  - *(iii) The probability p_{B-A} is more than the probabilities of overdemanded pairs that can be matched with a $B - A$ pair in larger exchanges: $p_{B-A} > \sum_{X-Y\in P^{O}(B-A)} p_{X-Y}$.*

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Dynamically efficient mechanism of a single hospital

- A dynamic kidney exchange mechanism matches incompatible pairs at each time $t \geq 0$.
- The time between two arrivals at a hospital h_i, denoted by τ, has an exponential distribution with parameter λ_{h_i}.
- The probability density function of which is $\lambda_{h_i} e^{-\lambda_{h_i}\tau}$.
- Pairs arrive at hospitals according to a stochastic process and possibly, they wait to be matched. Waiting is costly.
- Assume a fixed monetary cost c per unit time of waiting. The time discount rate is ρ.
- Suppose h_i implements a matching mechanism φ_i. The set of pairs matched by φ_i under the flow A_i until time t is denoted by $\varphi_i(t, A)$.
- $n^{\varphi_i}(\tau, A)$ is the number of matched recipients at time τ under φ_i.
- The *present and future exchange surplus* is given as

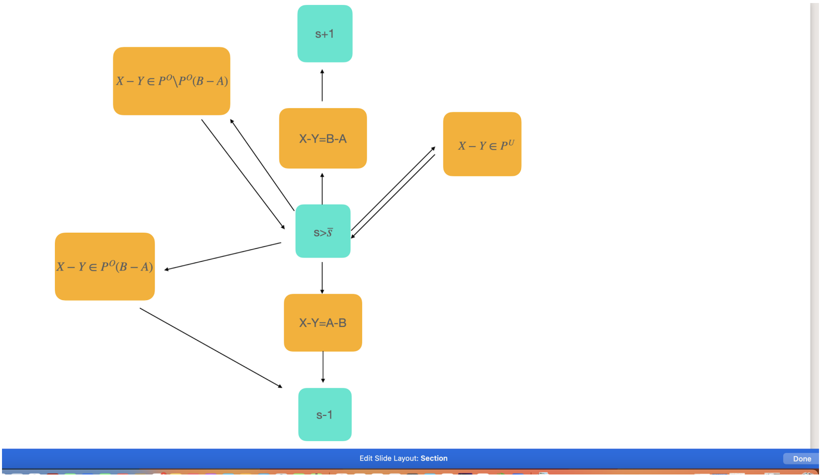
$$\widetilde{\mathbb{ES}}_i^{\varphi_i}(t) = \frac{c n_i^{\varphi_i}(t, A_i)}{\rho} + \int_{t^+}^{\infty} c \left(\mathbb{E}_t \left[\mid \varphi_i(\tau, A_i) \mid - \mid \varphi_i(t, A_i) \mid\right]\right) e^{-\rho(\tau - t)} d\tau. \quad (2)$$

Proposition

Suppose Long Run Assumption holds. If for all $A-B$ and $B-A$ pairs, p_{A-B} and p_{B-A} are sufficiently close to each other, then under any dynamically efficient multi- way matching mechanism, overdemanded type pairs are matched as soon as they arrive at the exchange pool.

- A matching mechanism is **dynamically efficient** if for each t, it maximizes the sum of the present and future exchange surplus at time t.
- The state of h_i is represented by $s_{h_i} = \mid B-A \mid - \mid A-B \mid$.
- If the number of accumulated $B-A$ pairs is less than or is equal to the threshold, then the arriving *overdemanded* pair is matched with an *underdemanded* pair in a two-way exchange (without including a $B-A$ pair). Due to $B-A$ pairs are not matched at this point and reserved for future exchanges, we call this action 'do-not-match' and denote it as **DNM**.
- If the number of accumulated $B-A$ pairs is greater than the threshold, then the arriving *overdemanded* pair is matched with a $B-A$ pair and an *underdemanded* pair in a three-way exchange. We call this action as 'match' and denote it as **M**.
- A Markov mechanism with a threshold $\bar{s}$, denoted by $\varphi_i^{\bar{s}}$, is formally defined as follows:

$$\varphi_i^{\bar{s}}(s_{hi}) = \begin{cases} DNM, & \text{if} \quad 0 \leq s_{h_i} \leq \bar{s} \\ M, & \text{otherwise} \end{cases}.$$


s+1
$X-Y \in P^O \backslash P^O(B-A)$
X-Y=B-A
$X-Y \in P^U$
s>$\bar{s}$
$X-Y \in P^O(B-A)$
X-Y=A-B
s-1
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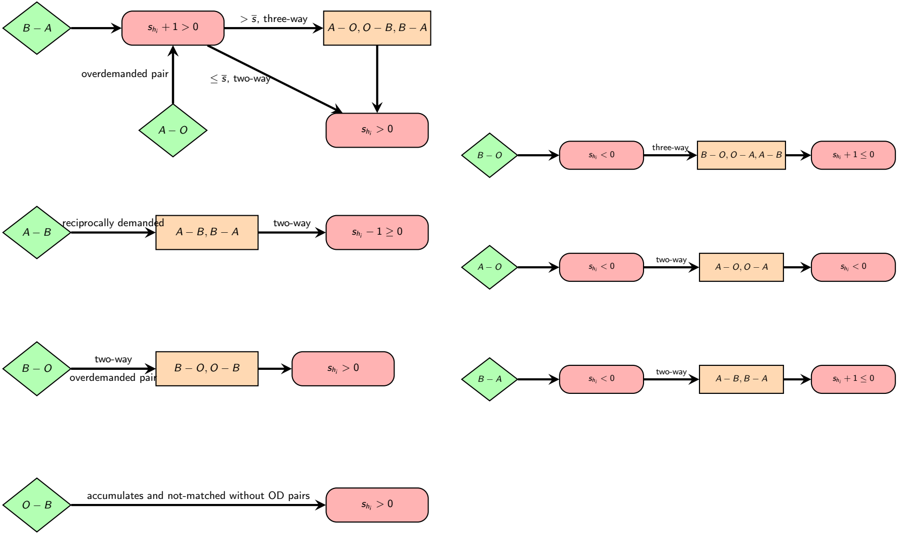
How does $\varphi_i^{\bar{s}}$ work?



- The time between two arrivals at h_i, denoted by τ_i, has an exponential distribution with parameter λ_{h_i}.
- The probability density function of which is $\lambda_{h_i} e^{-\lambda_{h_i}\tau_i}$.
- $\Longrightarrow$ The expected total discounting for a new pair's arrival is given by

$$\mathbb{E}\left[e^{-\rho\tau_i}\right] = \int_0^\infty \lambda_{h_i} e^{-\rho\tau_i} e^{-\lambda_{h_i}\tau_i} d\tau_1 = \frac{\lambda_{h_i}}{\lambda_{h_i} + \rho}.$$

- When a pair is matched in the exchange pool, the surplus related to the pair is $\frac{c}{\rho}$.
- $\mathbb{ES}(s)$ the total surplus at state $s \in S$ under the efficient rule is the sum of surpluses regarding each type:

$$\mathbb{ES}^i(s) = \sum_{X-Y\in\mathcal{P}^O} \mathbb{ES}^i_{X-Y}(s) + \mathbb{ES}^i_{A-B}(s),$$

- For each $X - Y \in \mathcal{P}^O$ the exchange surplus $\mathbb{ES}^i_{X-Y}(s) = \left(\frac{\lambda_{h_i} p_{X-Y}}{\rho}\right) 2\frac{c}{\rho}$. **One *underdemanded* pair and one *overdemanded* pair can be matched without using the *reciprocally demanded* pairs and *underdemanded* pairs are not matched without using an *overdemanded* pair**.
- When state of the hospital $s > 0$, the surplus related to reciprocally demanded pairs:

$$\mathbb{ES}^i_{B-A}(s) = \frac{\lambda_{h_i}}{\lambda_{h_i} + \rho} \left[\left(\sum_{X-Y \in \mathcal{P}^O \setminus \mathcal{P}^O(B-A) \cup \mathcal{P}^U} p_{X-Y} \right) \mathbb{ES}^i_{B-A}(s) + \left(\sum_{X-Y \in \mathcal{P}^O(B-A)} p_{X-Y} \right) \max\{\mathbb{ES}^i_{B-A}(s), \mathbb{ES}^i_{B-A}(s-1) + \frac{c}{\rho}\} + (p_{B-A})\, \mathbb{ES}^i_{B-A}(s+1) + (p_{A-B}) \left(\mathbb{ES}^i_{B-A}(s-1) + 2\frac{c}{\rho} \right) \right]. \quad (3)$$

Theorem (Unver,2010)

There exists $\bar{s} \geq 0$ *such that the Markov mechanism with threshold* $\bar{s}$ *is dynamically efficient.*

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Pooling

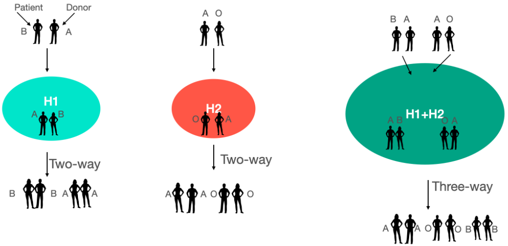
- The time between two arrivals at h_i, denoted by τ_i, has an exponential distribution with parameter λ_{h_i}, the probability density function of which is $\lambda_{h_i} e^{-\lambda_{h_i}\tau_i}$.
- Pairs arrive at the combined pool with a Poisson process of the arrival rate $\lambda_{h_1} + \lambda_{h_2}$.
- The *dynamically efficient* mechanism for the comined pool is also a threshold Markov mechanism. We denote this mechanism by $\varphi_p^{\overline{s_p}}$.
- For the combined exchange pool, the state is equal to $s_1 + s_2$
- The threshold value of the dynamically efficient mechanism for the combined pool: $\overline{s_p} > \overline{s}$. **The expected total discounting under pooling is** $\frac{2\lambda}{2\lambda+\rho} > \frac{\lambda}{\lambda+\rho}$.

Intuition

Three different effects on the exchange surplus under **full-centralization**:

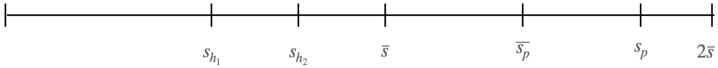
- Waiting is more costly under pooling:
- the arrival rate of pairs increases to 2λ and the expected time to wait for the first arrival is $\frac{1}{2\lambda}$.
- When a pair is matched in the exchange pool, the surplus related to the pair is $\frac{c}{\rho}$.
- The expected surplus of a new incoming pair under pooling is $\frac{2\lambda}{2\lambda+\rho}(\frac{c}{\rho}) < \frac{\lambda}{\lambda+\rho}(\frac{c}{\rho}) + \frac{\lambda}{\lambda+\rho}(\frac{c}{\rho})$, the expected surplus of a new incoming pair under matching it individually.
- Waiting costs are higher under pooling and any future exchange has a relatively lower value.
- The future exchanges are unable to offset the loss of surplus caused by the early matching of incoming pairs in the initial time periods under pooling mechanism.

- Waiting time is reduced due to inter-hospital exchanges under pooling:
- Pooling facilitates some immediate exchanges which might not be available when hospitals manage their incoming pairs separately.
- Pooling decreases waiting due these inter-hospital exchanges under pooling.



- $\frac{2\lambda}{2\lambda+\rho} > \frac{\lambda}{\lambda+\rho} \longrightarrow$ The value of matched $A-B$ and $B-A$ pairs in future periods increases $\longrightarrow \overline{s_p} \geq \overline{s}$.
- $\overline{s_p} > 2\overline{s}$ or $\overline{s_p} < 2\overline{s}$ is not known.
- This implies two different effects under pooling in terms of the total waiting time.

- If $\overline{s_p} < 2\bar{s}$, waiting is reduced in this case under pooling.
- If $s_{h_1} + s_{h_2} > \overline{s_p}$, then M is chosen in the combined pool and incoming $A - O$ to the h_2 is matched in a three-way exchange $A - O, O - B, B - A$.



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- This effect is reversed when the threshold under pooling becomes more than the double of the threshold of a single hospital, $\overline{s_p} > 2\overline{s}$. Assume that $s_{h_1} + s_{h_2} < \overline{s_p}$:

	hospital h_1 ($\varphi_1^{\overline{s}}$)	hospital h_2 ($\varphi_2^{\overline{s}}$)	pooling ($\varphi_P^{\overline{s_P}}$)
states at $t > 0$	$s_{h_1} > \overline{s}$	$0 \leq s_{h_2} < \overline{s}$	$s_{h_1} + s_{h_2} < \overline{s_P}$
actions	M	DNM	DNM
incoming pair type	$P^O(B-A)$	$P^O(B-A)$	$P^O(B-A)$
match at $\tau > t > 0$	$(OD - UD - (B-A))$	$(OD - UD)$	$(OD - UD)$

Table 2: Waiting time might increase under pooling.

- The incoming pairs do not belong to h_1 or h_2 anymore.
- If $s_{h_1} + s_{h_2} < \overline{s_p}$, then *DNM* is chosen in the combined pool and incoming $A-O$ to the h_1 is matched in two-way exchange $A-O$, $O-A$.

Theorem 2

Theorem

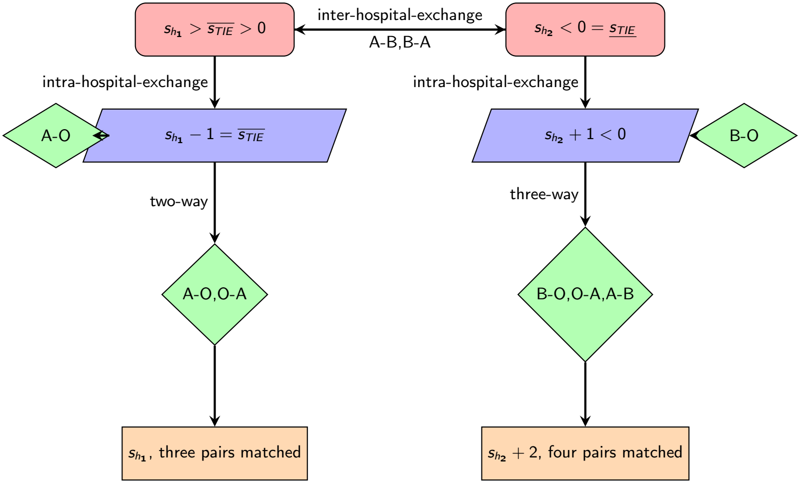
Suppose that, at time $t = 0$, states of both hospitals are zero. Then, there exists $\overline{\lambda}$ such that for each arrival rate $\lambda \leq \overline{\lambda}$, pooling reduces the exchange surplus compared to each hospital managing their arriving pairs separately.

- The upper bound for the arrival rate of pairs is $\overline{\lambda} = \sqrt{\frac{\rho}{p_{A-B}^2 - p_{B-A}^2}}$.
- The theorem demonstrates that state of both hospitals equal to zero and the arrival rate is sufficiently low, the negative effect (increased waiting costs plus increased waiting time in some situations) dominates the positive effect (reduced waiting time) under pooling.

Threshold with Inter-hospital Exchange

- The *TIE* is a threshold Markov mechanism.
- Given the number of reciprocally demanded pairs at the two hospitals, they decide on two things when an overdemanded pair arrives:
  1. They mutually engage into an *inter-hospital exchange* (*ihe*) or not (*nihe*).
  2. Then, each hospital decides whether to make a two-way (*DNM*) or a three-way exchange (*M*) with its own pairs depending on certain thresholds.
- Each pair $s = (s_{h_i}, s_{h_{-i}}) \in S = \mathbb{Z}^2$ defines a state.
- The thresholds of the *TIE* are $\overline{s_{TIE}}, \underline{s_{TIE}} \in \mathbb{Z}$ with $\overline{s_{TIE}} > 0, \underline{s_{TIE}} = 0$.
- Each hospital has a upper-threshold $\overline{s_{TIE}} > 0$, and a lower-threshold $\underline{s_{TIE}} = 0$. An *inter-hospital exchange* occurs, when there is at least one $A - B$ pair at h_{-i} and there are more $B - A$ pairs than the upper-threshold $\overline{s_{TIE}} > 0$ at h_i.

Example for the *TIE* mechanism



Formal Definition of TIE mechanism

The *TIE* is formally a Markov mechanism denoted by

$$\Phi_i^{\overline{s_{TIE}},\underline{s_{TIE}}} : S \longrightarrow \{ihe, nihe\} \times \{M, DNM\} \qquad (4)$$

for $i \in \{1,2\}$ such that $\Phi_i(s) = (ihe, x)$ if and only if $\Phi_{-i}(s) = (ihe, y)$ for each $x, y \in \{M, DNM\}$. We denote $\Phi(s) = \left(\Phi_1^{\overline{s_{TIE}},\underline{s_{TIE}}}, \Phi_2^{\overline{s_{TIE}},\underline{s_{TIE}}}\right)$.

- Both hospitals cannot choose M after an *inter-hospital* exchange $\longrightarrow s^{'}_{h_i} \nless 0,\ s^{'}_{h_{-i}} \ngtr \overline{STIE}$.

$$\Phi(s) = \begin{cases} (ihe,\ DNM), (ihe,\ M) & \text{if} \quad s_{h_i} = -1, \quad s_{h_{-i}} > \overline{STIE} + 1 \\ (ihe,\ M), (ihe,\ DNM) & \text{if} \quad s_{h_i} < -1, \quad s_{h_{-i}} = \overline{STIE} + 1 \\ (ihe,\ DNM), (ihe,\ DNM) & \text{if} \quad s_{h_i} = -1, \quad s_{h_{-i}} = \overline{STIE} + 1 \\ (nihe,\ M), (nihe,\ DNM) & \text{if} \quad s_{h_i} < 0, \quad 0 \leq s_{h_{-i}} \leq \overline{STIE} \\ (nihe,\ M), (nihe,\ DNM) & \text{if} \quad \overline{STIE} < s_{h_i}, \quad 0 \leq s_{h_{-i}} \leq \overline{STIE} \\ (nihe,\ DNM), (nihe,\ DNM) & \text{if} \quad \overline{STIE} \geq s_{h_i} > 0, \quad \forall i \in \{1,2\} \\ (nihe,\ M), (nihe,\ M) & \text{if} \quad s_{h_i} < 0, \quad \forall i \in \{1,2\} \end{cases}.$$

- Let $p\left(s_{h_i} > \overline{s_{TIE}}\right) = \theta$ and $p\left(s_{h_{-i}} < 0\right) = \gamma$. The arrival rates at h_i is equal to

$$\lambda'_{h_i} = \theta\gamma\left(\lambda + \lambda p_{A-B}\right) + \left(1 - \theta\gamma\right)\lambda = \lambda + \left(\theta\gamma\lambda p_{A-B}\right).$$

- The arrival rates at h_{-i} is equal to

$$\lambda'_{h_{-i}} = \theta\gamma\left(\lambda + \lambda p_{B-A}\right) + \left(1 - \theta\gamma\right)\lambda = \lambda + \left(\theta\gamma\lambda p_{B-A}\right).$$

- The time between two arrivals at h_i, denoted by τ_i, has an exponential distribution with parameter λ'_{h_i}, the probability density function of which is $\lambda'_{h_i} e^{-\lambda'_{h_i}\tau_i}$.

$$\mathbb{E}\left[e^{-\rho\tau_i}\right] = \int_0^\infty \lambda'_{h_i} e^{-\rho\tau_i} e^{-\lambda'_{h_i}\tau_i} d\tau_i = \frac{\lambda'_{h_i}}{\lambda'_{h_i} + \rho}. \tag{5}$$

- The expected total discounting is equal to $\frac{\lambda'}{\lambda' + \rho} > \frac{\lambda}{\lambda + \rho} \longrightarrow$ the new upper-threshold $\uparrow \longrightarrow \overline{s_{TIE}}$.
- Each hospital matches as much as possible $A - B$ pairs, the more frequently arrived pair. The lower-threshold of TIE is equal to $\underline{s_{TIE}} = 0$.

total exchange surplus

- The states of both hospitals are updated when they make an *inter-hospital exchange* and match one of their $B - A$ pairs with one of $A - B$ pairs $\longrightarrow$ the surplus of the hospital with the state s_{h_i} also depends on the state of the other hospital
- If there are *no-inter-hospital exchange*, the surplus of each hospital is determined regarding only its own state through an *intra-hospital* exchange.

The expected exchange surplus of h_i obtained from the *reciprocally demanded* pairs is equal to the following equation:

$$\mathbb{ES}^{i}_{A-B}\left(s_{h_i}, s_{h_{-i}}\right) = \frac{\lambda'_{h_i}}{\lambda'_{h_i} + \rho}\Bigg[(1-\theta\gamma)\,\mathbb{ES}^{i}_{A-B}\left(s_{h_i}, s_{h_{-i}}\right) + (\theta\gamma)\left(\mathbb{ES}^{i}_{A-B}\left(s_{h_i}-1, s_{h_{-i}}+1\right) + \frac{c}{\rho}\right)\Bigg]. \quad (6)$$

Proposition

For any reciprocally demanded pairs and for each hospital h_i where $i \in \{1, 2\}$, there exists a unique solution $\mathbb{ES}^{i}_{A-B} : \mathbb{Z}^2 \longrightarrow \mathbb{R}_+$ to the Bellman Equation (6).

Theorem

(a) *Under Assumptions 1-5, each hospital obtains higher surplus by implementing the TIE mechanism than the dynamically efficient exchange mechanism* $\varphi_i^{\overline{s}}$. *Under violation of Assumption 3, there exists* $\overline{\lambda}$ *such that for each arrival rate* $\lambda \leq \overline{\lambda}$, *each hospital obtains higher surplus by implementing the TIE mechanism than the dynamically efficient exchange mechanism* $\varphi_i^{\overline{s}}$.

(b) *There exists* $\overline{\overline{\lambda}}$ *such that for each arrival rate* $\lambda \leq \overline{\overline{\lambda}}$, *each hospital obtains higher surplus by implementing the TIE mechanism than the efficient mechanism* $\varphi_P^{\overline{sP}}$.

- The threshold inter-hospital-exchange mechanism aims the same goal through benefiting from the difference between the probabilities p_{A-B} and p_{B-A} and making *inter-hospital* exchanges between $A-B$ and $B-A$ pairs of the hospitals.
- The reciprocally demanded pairs are also matched in *inter-hospital-exchanges* through the *TIE* mechanism. This increases the number of two-way exchanges between $A-B$ and $B-A$.
- The importance of the *inter-hospital-exchanges* comes out for the arrival rates of pairs smaller than a specific upper-bound. When $p_{A-B} - p_{B-A} \uparrow \Longrightarrow$, $\overline{\lambda} \downarrow \Longrightarrow$ the upper-threshold of the *TIE* $\downarrow \Longrightarrow$ the probability of making an *inter-hospital-exchange*, $\theta\gamma \uparrow \Longrightarrow$ the surplus obtained from two-way *inter-hospital exchanges* become more of an issue and we need an upper-bound for the arrival rate of pairs to increase the probability of making an inter-hospital exchange.

- For the time discount, ρ is fixed
- Working with smaller arrival rates $\lambda < \overline{\lambda}$, hospitals make more frequently *inter-hospital-exchanges* and the harm of waiting longer for making a three-way exchanges in an *intra-hospital exchange* is compensated by frequently made two-way *inter-hospital exchanges.*
- By Assumption $p_{B-A} > \sum_{X-Y \in P^{O}(B-A)} p_{X-Y} \implies$ B-A pairs can be involved in a three-way exchange less frequently. The arrival rate of overdemanded pairs is less than the arrival rate of B-A. Under *TIE* mechanism, more number of B-A pairs can be matched with A-B pairs in *inter-hospital-exchanges*, despite hospitals wait longer to make an *inter-hospital-exchange* ($\overline{s_{TIE}} > \overline{s}$)

Thank You

Ünver, M. U. (2010): "Dynamic kidney exchange," *The Review of Economic Studies*, 77(1), 372–414.