The Semi-random Process

(The DEs method in action)

Paweł Prałat

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Department of Mathematics, Toronto Metropolitan University

Definition of the Process

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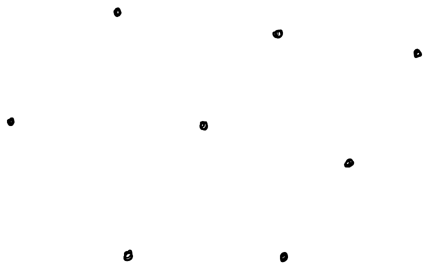

The semi-random graph process is a single player game in which the player is initially presented the empty graph G_0 on the vertex set $[n] := \{1, \ldots, n\}$.

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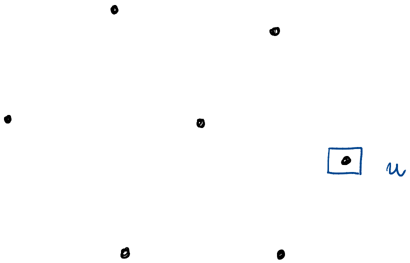

In each round $t \geq 1$, a vertex u_t (square) is drawn independently and u.a.r. (uniformly at random) from $[n]$ and then presented to the player.

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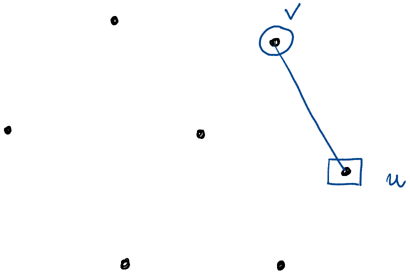

The player then adaptively selects a vertex v_t (circle), and adds the edge $u_t v_t$ to G_{t-1} to form the graph G_t.

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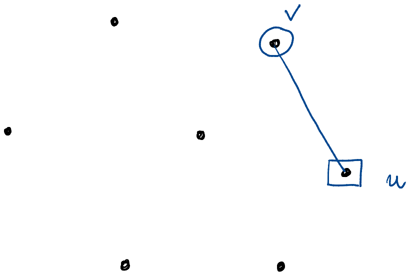

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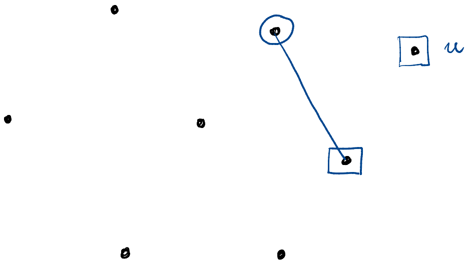

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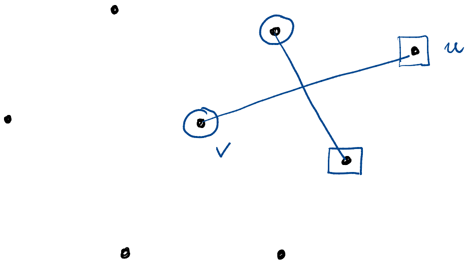

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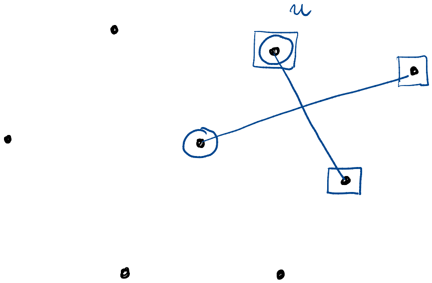

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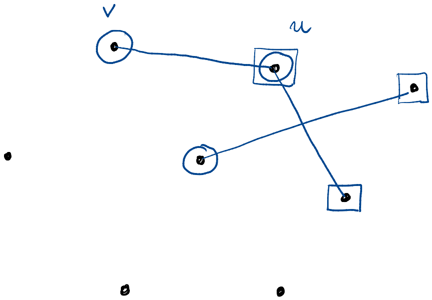

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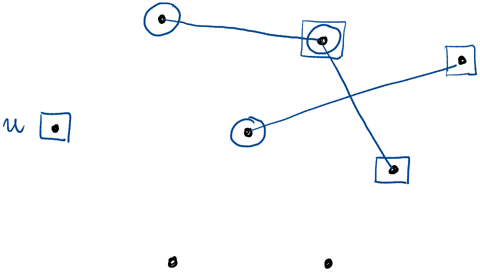

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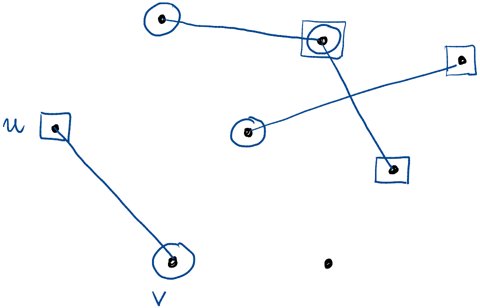

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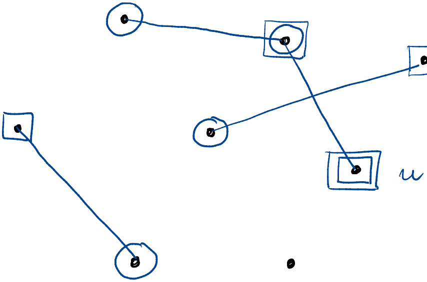

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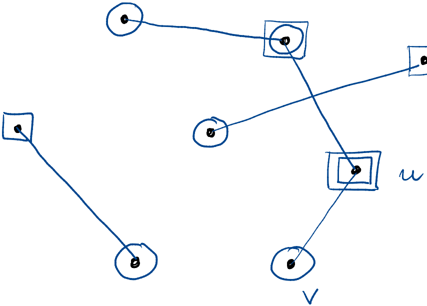

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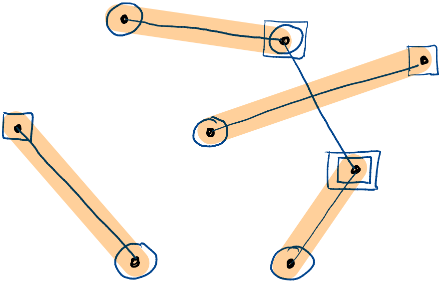

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Event $\mathscr{E} = (\mathscr{E}_n)_{n\geq 1}$ holds asymptotically almost surely (a.a.s.) if the probability that $\mathscr{E}_n$ holds tends to 1 as $n \to \infty$.

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Upper Bounds: Show that there exists a strategy $\mathcal{S}$ and a function $\tau = \tau(n)$, such that $G_\tau^{\mathcal{S}}$ satisfies $\mathscr{P}$ a.a.s.

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Thus, for a specific graph H_n (such as a perfect matching, or Hamiltonian cycle), the goal is to find the optimal (linear-time) strategy.

Summary of Results

Summary – Perfect Matchings

Perfect Matchings:

– upper bound $1.73576n$ improved to $1.20524n$.

– lower bound $0.69314n$ improved to $0.93261n$.

(Gao, MacRury, Prałat, SIDMA, 2022)

Summary – Hamilton Cycles

Hamilton Cycles:

– upper bound $3n$ improved to $2.61135n$.

– lower bound $1.21973n$ improved by ϵn.

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Small Subgraphs:

– construct fixed graph G of degeneracy d.

– upper bound $n^{(d-1)/d}\omega$ (Ben-Eliezer et al., RSA, 2020).

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– lower bound $n^{(d-1)/d}/\omega$ for any graph G.

– generalization to hypergraphs.

– tight results for 1 square and any number of circles
(Behague, Marbach, Prałat, Ruciński, ArXiv, 2021+)

– many open questions are left for at least 2 squares!
(Behague, Prałat, Ruciński, 2023++)

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– (large) complete graphs, independent sets, chromatic number
(Gamarnik, Kang, Prałat, ArXiv, 2023+)

Summary – Generalizations

– Hypergraphs

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– "power of k choices"
(Prałat, Singh, ArXiv, 2023+)

Perfect Matchings

Emulating k-out Process

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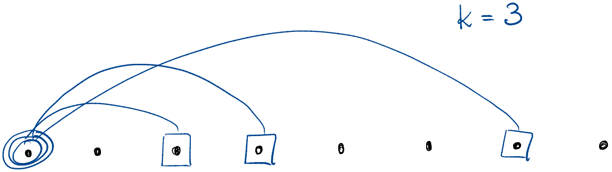
There exists a strategy such that a.a.s. $H(k) \subseteq G_{kn+\omega}$.
(Ben-Eliezer, Hefetz, Kronenberg, Parczyk, Shikhelman, and Stojaković, RSA, 2020)



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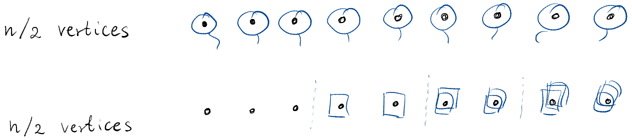

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Implication: a.a.s. there exists a strategy to create a perfect matching in $(1 + 2/e + o(1))n < 1.73576n$ rounds.

Perfect Matchings: Upper Bounds

A.a.s. there exists a strategy to create a perfect matching in $(\beta + 10^{-5})n \leq 1.20524n$ rounds, where β is derived from a system of differential equations.
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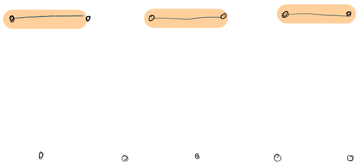

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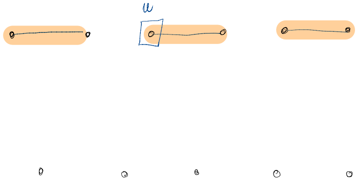

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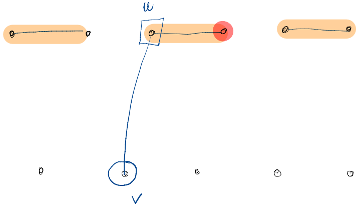

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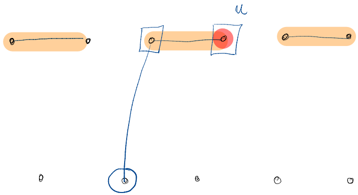

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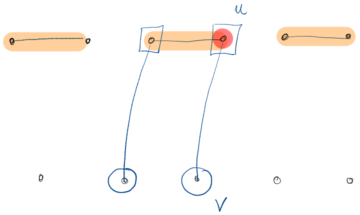

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– Our deterministic algorithm keeps all extensions, has k deterministic greedy phases for $k \ge 1100$, and concludes by executing the randomized algorithm.

Analysing the Randomized Algorithm

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We condition on less information so that the circle placements amongst the unsaturated vertices remain u.a.r.

Deriving the Differential Equations

$$\mathbb{E}[X(t+1) - X(t) \mid H_t] \quad = \quad \frac{2(n - X(t) + R(t))}{n} + O(1/n),$$

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\begin{aligned}
\mathbb{E}[X(t+1) - X(t) \mid H_t] &= \frac{2(n - X(t) + R(t))}{n} + O(1/n), \\
\mathbb{E}[R(t+1) - R(t) \mid H_t] &= \frac{n - X(t)}{n} \cdot \frac{-2R(t)}{n - X(t)} \\
&\quad + \frac{R(t)}{n}\left(-1 - \frac{2(R(t)-1)}{n - X(t)}\right) \\
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By writing $x(s) = X(sn)/n$ and $r(s) = R(sn)/n$ for $s \in [0, \infty)$, we have that

$$
\begin{aligned}
x' &= 2(1 - x + r), \\
r' &= \frac{-2r}{1 - x}(1 - x + r) - r + x - 2r,
\end{aligned}
$$

with the initial conditions $x(0) = r(0) = 0$.

Applying the DE Method

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The remaining $o(n)$ unsaturated vertices are matched via a clean-up algorithm which is analysed by a (lossy) elementary analysis.

Perfect Matchings: Lower Bounds

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A.a.s. no strategy can create a perfect matching in less than $(\ln(2) + o(1))n \geq 0.69314n$ rounds.
(Ben-Eliezer, Hefetz, Kronenberg, Parczyk, Shikhelman, and Stojaković, RSA, 2020)

Perfect Matchings: Lower Bounds

Let

$$\alpha = \inf\{b \geq 0 : g(b) \geq 1/2\},$$

where

$$g(b) := 1 + \frac{1-2b}{2}\exp(-b) - (b+1)\exp(-2b) - \frac{1}{2}\exp(-3b).$$

Then, a.a.s. no strategy can create a perfect matching in less than $(\alpha + o(1))n \geq 0.93261n$ rounds.
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We again use the DE method, though we must restrict to "well-behaved" strategies.

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Solution: We offer a deal the player will gladly accept:
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– Create a matching consisting of $n/2 - n/\omega$ edges in at most n rounds.
– Never create multi-edges.

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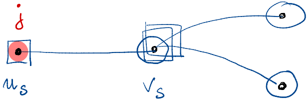
Vertex j is redundant at time $t \geq 0$ if:

– j is covered by precisely one square, say u_s for $s \leq t$,

– circle v_s connected to u_s by the player is covered by at least one square, which arrives after round s.



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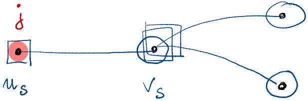

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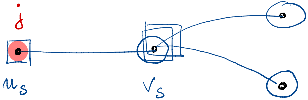

$U(t)$: the number of redundant vertices at time t.

$U(t)$ depends only on the placement of the squares, not the strategy.

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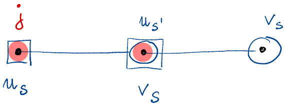

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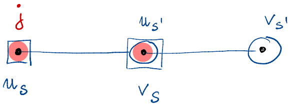

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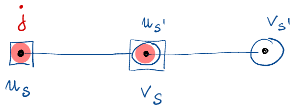

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To get the lower bound, we use the following inequality:

$$X(t) - U(t) + W(t) \ge \frac{n}{2} - \frac{3t}{\omega},$$

and the DE method of Nick Wormald.

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Implication: a.a.s. there exists a strategy to create a perfect matching in $(3 + o(1))n$ rounds.

There exists a strategy to create a Hamilton cycle in βn rounds a.a.s., where β is the result of a high dimensional optimization problem. Numerical computations indicate that $\beta < 2.61135$.
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One more trick brings it down to to $1.81696n$.
(Gao, Frieze, MacRury, Prałat, Sorkin, 2023+)

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We recently improved this bound to $1.26575n$ using similar techniques as in the perfect matching problem.

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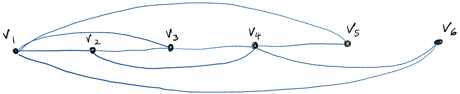

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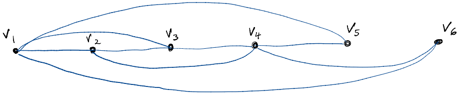

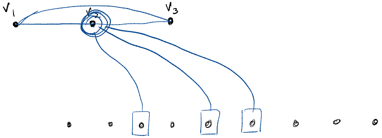

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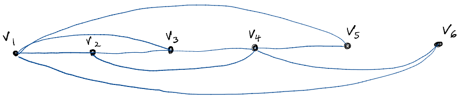

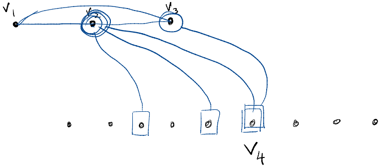

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Conjecture: a.a.s. no strategy can create G in $n^{(d-1)/d}/\omega$ rounds, where $\omega = \omega(n) \to \infty$ as $n \to \infty$.

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The semi-random process can be generalized to hypergraphs and some results can be transferred. But some questions are still open, for example, $G = K_6^{(3)}$ and 2 squares (1 circle).

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A starting point may be to consider when some additional structure is assumed to hold on H – i.e., it is vertex transitive, or at least Δ-regular.

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