# Singular cscK metrics on smoothable varieties

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# Outline

1. CscK problem
2. Mildly singular varieties
3. Main results

# Constant scalar curvature Kähler (cscK) metrics

Let $(X, \omega)$ be a compact Kähler manifold with $\dim_{\mathbb{C}} X = n$.
Locally,

$$\omega \underset{loc}{=} \sum_{\alpha,\beta} \omega_{\alpha\bar{\beta}} \, \mathrm{i} \, \mathrm{d}z_\alpha \wedge \mathrm{d}\bar{z}_\beta$$

where $(\omega_{\alpha\bar{\beta}})$ is hermitian, positive-definite and $\mathrm{d}\omega = 0$.

- $\mathrm{Ric}(\omega) \underset{loc}{=} -\mathrm{dd^c} \log(\det(\omega_{\alpha\bar{\beta}}))$ = Ricci form/curvature
  $\hookrightarrow$ globally defined $(1,1)$-form representing $c_1(X) \in H^{1,1}(X, \mathbb{R})$
- $\mathrm{S}(\omega) = \mathrm{tr}_\omega \mathrm{Ric}(\omega) = \frac{n \, \mathrm{Ric}(\omega) \wedge \omega^{n-1}}{\omega^n}$ = scalar curvature

A metric $\omega$ is called a cscK metric if $\mathrm{S}(\omega) = \bar{s} := \frac{n c_1(X) \cdot [\omega]^{n-1}}{[\omega]^n}$.

Example: Kähler–Einstein metrics (i.e. $\mathrm{Ric}(\omega) = \lambda\omega$) and their products

# CscK problem

Question: Can one find a cscK metric in a Kähler class $\alpha$?

Fix a Kähler metric $\omega \in \alpha$.

**Lemma ($\partial\bar{\partial}$-lemma)**

*$\omega' \in \alpha \iff \omega' = \omega + dd^c u$ for some $u \in \mathcal{C}^\infty(X)$.*

The space of Kähler potentials:

$$\mathcal{H}_\omega(X) := \{u \in \mathcal{C}^\infty(X) \,|\, \omega_u := \omega + \mathrm{dd^c} u > 0\}.$$

$\hookrightarrow \mathcal{H}_\omega/\mathbb{R} \simeq \{\omega' \text{ Kähler metric in } \alpha\}$.

The cscK equation $\mathrm{S}(\omega_u) = \bar{s}$ is the Eular–Lagrange equation of the Mabuchi functional (K-energy) $\mathbf{M} : \mathcal{H}_\omega \to \mathbb{R}$.

# Mabuchi functional

Mabuchi functional (Chen–Tian formula): for every $u \in \mathcal{H}_\omega$,

$$\mathbf{M}(u) = \mathbf{H}(u) + \bar{s}\,\mathbf{E}(u) - n\,\mathbf{E}_{\mathrm{Ric}(\omega)}(u).$$

- The entropy $\mathbf{H} : \mathcal{H}_\omega \to \mathbb{R}_{\geq 0}$ (leading term) is defined as

$$\mathbf{H}(u) = \frac{1}{V}\int_X \log\left(\frac{\omega_u^n}{\omega^n}\right)\omega_u^n.$$

- The energy functional $\mathbf{E} : \mathcal{H}_\omega \to \mathbb{R}$ (distance) is a primitive of $u \mapsto \mathrm{MA}(u) = \omega_u^n/V$; precisely,

$$\mathbf{E}(u) = \frac{1}{(n+1)V}\sum_{j=0}^{n}\int_X u\,\omega_u^j \wedge \omega^{n-j}, \quad \frac{\mathrm{d}}{\mathrm{d}t}\mathbf{E}(u_t) = \int_X \dot{u}_t\frac{\omega_{u_t}^n}{V}.$$

- The $\Theta$-energy $\mathbf{E}_\Theta(u) = \frac{1}{nV}\sum_{j=0}^{n-1}\int_X u\,\Theta \wedge \omega_u^j \wedge \omega^{n-1-j}$.

# Distance on $\mathcal{H}_\omega$

$d_1$-distance: consider $(u_t)_{t\in[0,1]}$ a smooth curve in $\mathcal{H}_\omega$ joining $u_0$ & $u_1$,

$$d_1(u_0, u_1) := \inf_{u_t} \int_0^1 \left( \int_X |\dot{u}_t| \frac{\omega_{u_t}^n}{V} \right) \mathrm{d}t.$$

$\mathbf{E}$ is monotone increasing $\rightsquigarrow$ unique extension to all $u \in \mathrm{PSH}(X, \omega)$

$$\mathbf{E}(u) := \inf \{\mathbf{E}(v) \,|\, u \leq v \in \mathcal{H}_\omega\} \in [-\infty, \infty).$$

The finite energy class (Guedj–Zeriahi '07) is defined as

$$\mathcal{E}^1(X, \omega) := \{u \in \mathrm{PSH}(X, \omega) \,|\, \mathbf{E}(u) > -\infty\}.$$

**Theorem (Darvas '15)**

- $(\mathcal{E}^1(X, \omega), d_1)$ *is a metric completion of* $(\mathcal{H}_\omega, d_1)$*;*
- $d_1(u, v) = \mathbf{E}(u) + \mathbf{E}(v) - 2\,\mathbf{E}(P_\omega(u, v))$*;*
- $(\mathcal{E}^1(X, \omega), d_1)$ *is a geodesic metric space.*

# Convexity, minimizer & existence characterization

**Theorem**

- **M** *is convex along geodesics in* $\mathcal{E}^1(X, \omega)$ *(Berman–Berndtsson '17, Berman–Darvas–Lu '17);*
- *Find a minimizer* $\varphi$ *of* **M** $\iff$ *find a cscK* $\omega_\varphi$ *in* $[\omega]$ *(Darvas–Rubinstein '17, Berman–Darvas–Lu '20, Chen–Cheng '21).*

**Theorem (DR'17, BDL'20, CC'21)**

*TFAE*

1. *There exists a unique cscK metric* $\omega_{\mathrm{cscK}} \in [\omega]$*;*
2. **M** *is coercive, i.e.* $\exists\, A > 0$ *and* $B > 0$ *such that*

$$\mathbf{M}(u) \geq A(-\mathbf{E}(u)) - B, \quad \forall u \in \mathcal{E}^1_{\mathrm{norm}}(X, \omega).$$

# CscK equations and YTD conjecture

Finding a cscK metric in $[\omega]$ boils down to solving the following couple of equations with unknown pair $(\varphi, F)$,

$$(\omega + dd^c\varphi)^n = e^F \omega^n \quad \text{and} \quad \Delta_{\omega_\varphi} F = -\bar{s} + \mathrm{tr}_{\omega_\varphi} \mathrm{Ric}(\omega).$$

Chen–Cheng '21: $\|\varphi\|_{L^\infty} + \|F\|_{L^\infty} \leq C(\mathbf{H}(\varphi), X, \omega)$.

**Yau–Tian–Donaldson Conjecture**

$\exists!$ *cscK in* $[\omega] \iff (X, [\omega])$ *is uniformly K-stable*

Rmk: $X$ Fano, $\exists!$ KE in $c_1(-K_X) \Leftrightarrow (X, -K_X)$ is K-stable (CDS'15, ...)

$\exists!$ cscK in $[\omega]$ $\Rightarrow$ **M** is coercive (DR'17, BDL'20)

**M** is coercive $\Rightarrow$ $\exists!$ cscK in $[\omega]$ (CC'21)

**M** is coercive $\Rightarrow$ uniform K-stability (BHJ'19, DR'17, SD'20)

uniform K-stability $\dashrightarrow$ **M** is coercive (C.Li'22, Mesquita-Piccione'24)

# *Singular setting*

# Mildly singular varieties

Why singular varieties/degenerate families ?

- Minimal Model Program, Ueno '75 $\exists$ 3-fold has no smooth minimal model
- Compactifying moduli spaces, e.g. $r \ll 1$

$$\mathcal{X} = \{([z], t) \in \mathbb{P}^3 \times \mathbb{D}_r \mid z_1 z_2 z_3 + z_0^3 - t \sum_{i=1}^3 z_i^3 = 0\} \xrightarrow{\pi = \mathrm{pr}_2} \mathbb{D}_r$$

By variety, we mean an irreducible reduced complex analytic space.

Assume that $X$ is a $\mathbb{Q}$-Gorenstein variety, i.e.

- $X$ is a normal variety.
  $\hookrightarrow$ $\operatorname{codim} X^{\mathrm{sing}} \geq 2$ and $X$ is locally irreducible.
- $K_X$ is a $\mathbb{Q}$-line bundle.
  Namely, $\exists\ m \in \mathbb{N}^*$ and a line bundle $L$ on $X$ s.t $L_{|X^{\mathrm{reg}}} = mK_{X^{\mathrm{reg}}}$.

# Adapted measures and klt singularities

What are cscK metrics on singular varieties?

Adapted measure: Let $h$ be smooth hermitian metric on $mK_X$,

$$\mu_h := \mathrm{i}^{n^2} \left( \frac{\Omega \wedge \overline{\Omega}}{|\Omega|_h^2} \right)^{1/m} \quad \text{where } \Omega\text{: local generator of } mK_X.$$

CscK pbm makes sense if $X$ has Kawamata log terminal (klt) singularities.

$X$ is klt $\iff$ $\mu$ has finite mass near $X^{\mathrm{sing}}$.

- klt $\iff \forall r : Y \to X$ resol'n, $K_Y = r^* K_X + \sum_i a_i E_i$, $\forall a_i > -1$.
- In particular, $\mu = f\omega^n$ where $f \in L^p(X, \omega^n)$ for some $p > 1$.
- E.g.: ordinary double point ($\sum_i x_i^2 = 0$), quotient singularities, etc
- Odaka '13: K-semistable $\mathbb{Q}$-Fano $\implies$ at worst klt singularities

# Singular cscK metrics

The Ricci curvature of an adapted measure $\mu_h$ is defined by

$$\mathrm{Ric}(\mu_h) \underset{loc}{=} dd^c \log |\Omega|_h^{2/m} \in c_1(X).$$

Let $X$ be klt and fix $\omega$ a smooth Kähler metric on $X$.
We want to solve

$$\begin{aligned} (\omega + dd^c\varphi)^n &= e^F \mu \\ \Delta_{\omega_\varphi} F &= -\bar{s} + \mathrm{tr}_{\omega_\varphi} \Theta \end{aligned} \qquad \text{where } \Theta = \mathrm{Ric}(\mu).$$

The corresponding Mabuchi functional:

$$\mathbf{M}(u) := \mathbf{H}_\mu(u) + \bar{s}\,\mathbf{E}(u) - n\,\mathbf{E}_\Theta(u) - C$$

where $\mathbf{H}_\mu(u) := \frac{1}{V}\int_X \log\left(\frac{\omega_u^n}{\mu}\right)\omega_u^n$.

**Theorem (Di Nezza–Lu '22)**

$\mathbf{M}$ *is convex along geodesics in* $\mathcal{E}^1(X,\omega)$.

# Main results

Setup:

- $\mathcal{X}$ is an $(n+1)$-dimensional $\mathbb{Q}$-Gorenstein variety
- $\pi : \mathcal{X} \to \mathbb{D}$ is a proper holo. surj. map, and $X_t := \pi^{-1}(t)$ normal $\forall t$
- $\omega$ a hermitian metric, relatively Kähler on $\mathcal{X}$ ($\omega_t := \omega_{|X_t}$ Kähler $\forall t$)

**Main Theorem (P.–Tô–Trusiani '23)**

*Under the above setting, suppose that $X_0$ is klt.*

(1) *The coercivity threshold*

$$\sigma(t) := \sup\{A \in \mathbb{R} \mid \exists B \in \mathbb{R} \; s.t.\, \mathbf{M}_t \geq A(-\mathbf{E}_t) - B \text{ on } \mathcal{E}^1_{\text{norm}}(X_t, \omega_t)\}$$

*is lower semi-continuous near* 0.

(2) *If $\mathcal{X}$ is a smoothing of $X_0$ and $\mathbf{M}_0$ is coercive, then $X_0$ admits a singular cscK metric $\omega_{0,\text{cscK}}$ in $[\omega_0]$.*

# Some developments and remarks

Openness under deformation: $X_0$ has !cscK in $[\omega] \Rightarrow$ so does $X_t$, $\forall t \sim 0$

- LeBrun–Simanca '94: $\pi : \mathcal{X} \to \mathbb{D}$ is smooth.
- Biquard–Rollin '15: $\pi : \mathcal{X} \to \mathbb{D}$ is a smoothing of $X_0$ & $\dim_{\mathbb{C}} X_t = 2$.

Openness in Kähler cone $\mathcal{K}_X$: LS '94: $X$ smooth

Blum–Liu–Xu'22: K-stability is Zariski open in proj flat fami of Fano var.
Smooth Odaka'13/ anal. pf $\mathcal{X} \to \mathbb{D}$: Spotti–Sun–Yao'16, P.–Trusiani'23

Remark: In our result,

(1) supports the openness of K-stability in a more general context.

(1) also works for changing Kähler classes in singular setting, i.e.

$$\sigma([\omega]) := \sup\{A \in \mathbb{R} \mid \exists B \in \mathbb{R} \text{ s.t. } \mathbf{M}_\omega \geq A(-\mathbf{E}_\omega) - B \text{ on } \mathcal{E}^1_{\mathrm{norm}}(X, \omega)\}$$

is l.s.c. for $[\omega] \in \mathcal{K}_X$

(2) is the first step towards the analytic YTD for singular klt varieties.

# Main input 1: strong convergence in families

Let $t_k \to 0$ as $k \to +\infty$. Denote $X_k := X_{t_k}$ and $\omega_k := \omega_{t_k}$.

Convergence in families (P.–Trusiani '23):

We say that $\varphi_k \in \mathrm{PSH}(X_k, \omega_k)$ converges to $\varphi_0 \in \mathrm{PSH}(X_0, \omega_0)$ if $\forall$ data: $U_0 \Subset X_0^{\mathrm{reg}}$, $\mathcal{U} \Subset \mathcal{X} \setminus \mathrm{sing}(\pi)$, and $F$ s.t.

$$\begin{array}{ccc} U_0 \times \mathbb{D} & \xrightarrow[\sim]{F} & \mathcal{U} \\ & {\scriptstyle \mathrm{pr}_2} \searrow \quad \swarrow {\scriptstyle \pi} & \\ & \mathbb{D} & \end{array} \quad , \quad \text{and} \quad F_0 = \mathrm{Id}_{U_0},$$

the sequence $F_k^* \varphi_k$ converges to $\varphi_0$ in $L^1(U_0)$.

This notion is well-defined, i.e. does not depend on the choice of $(U_0, F)$.

Strong convergence in families (P.–Tô–Trusiani '23):

$$\varphi_k \text{ converges to } \varphi_0 \text{ and } \mathbf{E}_k(\varphi_k) \to \mathbf{E}_0(\varphi_0).$$

# Main input 2: key relative properties in families

**Propositions (P.–Tô–Trusiani '23)**

*Under our setting of the main theorem:*

(1) *Strong compactness:*
*Let* $(u_k)_k \in \bigsqcup_k \mathcal{E}^1_{\mathrm{norm}}(X_k, \omega_k)$ *s.t.* $(\mathbf{H}_k(u_k))_k$ *is uniformly bounded.*
*Then* $\exists$ *a subsequence converging strongly to some* $u_0 \in \mathcal{E}^1(X_0, \omega_0)$*.*

(2) *Lower semi-continuity of Mabuchi functional:*
*If* $(u_k)_k \in \bigsqcup_k \mathcal{E}^1(X_k, \omega_k)$ *converges strongly to* $u_0 \in \mathcal{E}^1(X_0, \omega_0)$*, then*

$$\mathbf{M}_0(u_0) \le \liminf_{k \to +\infty} \mathbf{M}_k(u_k).$$

# Proof: L.s.c. of coercivity threshold

On $X_0$, $\exists$ constants $A_0 \in \mathbb{R}$, $B_0 > 0$ s.t.

$$\mathbf{M}_0(u) \geq A_0(-\mathbf{E}_0(u)) - B_0, \forall u \in \mathcal{E}^1_{\mathrm{norm}}(X_0, \omega_0).$$

**Theorem (uniform coercivity)**

*For any $A < A_0$, $\exists$ $B > 0$ and $r > 0$ s.t.*

$$\mathbf{M}_t \geq A(-\mathbf{E}_t) - B \quad \text{on } \mathcal{E}^1_{\mathrm{norm}}(X_t, \omega_t), \quad \forall |t| < r.$$

Suppose by contradiction, $\exists B_k \to +\infty$, $t_k \to 0$, $u_k \in \mathcal{E}^1_{\mathrm{norm}}(X_k, \omega_k)$ s.t.

$$\mathbf{M}_k(u_k) < A(-\mathbf{E}_k(u_k)) - B_k.$$

Note: $|\mathbf{E}_{\Theta,t}(w)| \leq C_1 |\mathbf{E}_t(w)|$, $\forall w \in \mathcal{E}^1_{\mathrm{norm}}(X_t, \omega_t)$, so

$$\mathbf{H}_k(u_k) + (\bar{s} + C_1)\mathbf{E}_k(u_k) \leq \mathbf{M}_k(u_k) < A(-\mathbf{E}_k(u_k)) - B_k.$$

Enlarge $C_1 = C_1(\bar{s}, A) \gg 1 \implies -\mathbf{E}_k(u_k) = d_1(0, u_k) =: d_k \to +\infty$.

Take $g_k(s)$ the unit-speed geodesic in $\mathcal{E}^1_{\mathrm{norm}}(X_k, \omega_k)$ joining 0 & $u_k$.

Set $v_k := g_k(D) \implies -\mathbf{E}_k(v_k) = D$.

By the convexity of $\mathbf{M}$,

$$\mathbf{M}_k(v_k) \leq \frac{D}{d_k}\mathbf{M}_k(u_k) + \frac{d_k - D}{d_k}\mathbf{M}_k(0) \leq \frac{D}{d_k}(Ad_k - B_k) \leq AD$$

$$\mathbf{M}_k(v_k) \geq \mathbf{H}_k(v_k) + (\bar{s} + C_1)\,\mathbf{E}_k(v_k) = \mathbf{H}_k(v_k) + (\bar{s} + C_1)(-D)$$

$\hookrightarrow \mathbf{H}_k(v_k) \leq C_2 D$.

(1) $\implies$ $v_k$ sub-converges strongly to some $v_0 \in \mathcal{E}^1(X_0, \omega_0)$

(2) $\implies$ $\mathbf{M}_0(v_0) \leq \liminf_k \mathbf{M}_k(v_k) \leq AD$

By the assumption on $X_0$,

$$A_0 D - B_0 = A_0(-\mathbf{E}_0(v_0)) - B_0 \leq \mathbf{M}_0(v_0) \leq AD.$$

Take $D = \frac{B_0}{A_0 - A} + 1 \implies$ Contradiction!

# Existence on $\mathbb{Q}$-Gorenstein smoothable varieties

Chen–Cheng '21 (Deruelle–Di Nezza '22, Guo–Phong '22):
a priori estimates on a fixed Kähler manifold $(X, \omega)$ mainly depend on:

(1) $\mathbf{H}(\varphi) \leq K_1$, (2) $|\mathrm{Ric}(\omega)| \leq K_2 \omega$,

(3) $\exists \alpha > 0$ s.t. $\int_X e^{-\alpha\psi} \omega^n \leq K_3$, $\forall \psi \in \mathrm{PSH}_{\mathrm{norm}}(X, \omega)$.

Let $\varphi_t$ be a cscK potential on $\mathcal{E}^1_{\mathrm{norm}}(X_t, \omega_t)$.
Since $\mathbf{M}_0$ is coercive, the uniform coercivity gives

$$
\begin{aligned}
& 0 \geq \mathbf{M}_t(\varphi_t) \geq A(-\mathbf{E}_t(\varphi_t)) - B \\
& \implies -\mathbf{E}_t(\varphi_t) \leq B/A \\
& \implies \mathbf{H}_t(\varphi_t) \leq \mathbf{M}_t(\varphi_t) + (\bar{s} + C_1)(-\mathbf{E}_t(\varphi_t)) \leq C_2 B/A.
\end{aligned}
$$

Extracting a subsequential limit to a smooth potential $\varphi_0$ on $X_0^{\mathrm{reg}}$
$\hookrightarrow \varphi_0$ is a cscK potential !

*Thank you !*