Coupled KPZ equations and their decoupleability

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Recent trends in Stochastic Partial Differential Equations
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Coupled systems of interest

We consider coupled KPZ equations in $d = 1$ on $\mathbb{T} = [0, 1)$ or on $\mathbb{R}$ with periodic boundary condition:

$$\partial_i h^i = \frac{1}{2}\Delta h^i + \gamma^i_{j,k} \nabla h^j \nabla h^k + \xi^i_t$$

or their 'gradient' stochastic Burgers equations (SBE)

$$\partial_i u^i = \frac{1}{2}\Delta u^i + \gamma^i_{j,k} \nabla \left(u^j u^k\right) + \nabla \xi^i_t$$

–Here, we use the Einstein summation convention.

The coupling constant $\Gamma = \{\gamma^i_{j,k}\}$ is a real, $n \times n \times n$ tensor.

–Without loss of generality, it is always

Bilinear: By the symmetry say of the term $u^j u^k = u^k u^j$,

$$\gamma^i_{j,k} = \gamma^i_{k,j}.$$

–A stronger condition is that of

Trilinearity:

$$\gamma^i_{j,k} = \gamma^i_{k,j} = \gamma^j_{i,k},$$

sometimes also called 'completely symmetric'.

–Let $\mathcal{T}_n$ be the space of trilinear tensors.

Forms in $n = 2, 3$

$$\begin{pmatrix} \Gamma^1_{1,1} & \Gamma^1_{1,2} \\ \Gamma^1_{2,1} & \Gamma^1_{2,2} \end{pmatrix} = \begin{pmatrix} a_3 & a_2 \\ a_2 & a_1 \end{pmatrix} \text{ and } \begin{pmatrix} \Gamma^2_{1,1} & \Gamma^2_{1,2} \\ \Gamma^2_{2,1} & \Gamma^2_{2,2} \end{pmatrix} = \begin{pmatrix} a_2 & a_1 \\ a_1 & a_0 \end{pmatrix}.$$

$$\Gamma^1 = \begin{pmatrix} \Gamma^1_{11} & \Gamma^1_{12} & \Gamma^1_{13} \\ \Gamma^1_{21} & \Gamma^1_{22} & \Gamma^1_{23} \\ \Gamma^1_{31} & \Gamma^1_{32} & \Gamma^1_{33} \end{pmatrix} = \begin{pmatrix} a_1 & b_1 & b_3 \\ b_1 & b_2 & c \\ b_3 & c & b_4 \end{pmatrix}$$

$$\Gamma^2 = \begin{pmatrix} \Gamma^2_{11} & \Gamma^2_{12} & \Gamma^2_{13} \\ \Gamma^2_{21} & \Gamma^2_{22} & \Gamma^2_{23} \\ \Gamma^2_{31} & \Gamma^2_{32} & \Gamma^2_{33} \end{pmatrix} = \begin{pmatrix} b_1 & b_2 & c \\ b_2 & a_2 & b_5 \\ c & b_5 & b_6 \end{pmatrix}$$

$$\Gamma^3 = \begin{pmatrix} \Gamma^3_{11} & \Gamma^3_{12} & \Gamma^3_{13} \\ \Gamma^3_{21} & \Gamma^3_{22} & \Gamma^3_{23} \\ \Gamma^3_{31} & \Gamma^3_{32} & \Gamma^3_{33} \end{pmatrix} = \begin{pmatrix} b_3 & c & b_4 \\ c & b_5 & b_6 \\ b_4 & b_6 & a_3 \end{pmatrix}$$

Goals

We describe necessary/sufficient conditions on ‘trilinear’

$$\Gamma = \{\gamma^i_{j,k}\} \in \mathcal{T}_n$$

so that in an orthogonal frame σh or σu, where $\sigma \in O(n)$,
the system *decouples*, or *partially decouples*.
These are explicit when $n \leq 3$. BFSV 2025.

–Such a condition would characterize the fluctuation class of the system in terms of 1D classes, whether KPZ or EW.

–The main methods are via ‘invariant’ theory, which we will later discuss. Mathematica computations were needed.

–On the other hand, when the system does not decouple, there has been recent discussion of its ‘fluctuation class’,
not yet resolved....

Outline

We will begin with some of the related literature, before getting to the ‘decoupling’ problem.

Existence/Uniqueness

The KPZ/SBE system for $\{h^i\}$ or $\{u^i\}$ is ill-posed in the classical sense, requiring renormalization.

When Γ is bilinear,

Local-in-time well-posedness is shown by applying regularity structures or paracontrolled analysis.

–Hairer 2014, Gubinelli-Imkeller-Perkowski 2015.

However, when Γ is trilinear,

Global-in-time well-posedness holds with respect to initial values in a Hölder-Besov space $\left(\mathcal{C}^\alpha\right)^n$ on $\mathbb{T}$ for $\alpha < 1/2$.

As well, the SBE system has a unique invariant measure, namely $\mathbb{R}^n$-valued spatial white noise on $\mathbb{T}$.

–Funaki-Hoshino 2017.

Derivation from microscopic systems

One may derive SBE systems as limits of fluctuation fields of n types of particles interacting on $\mathbb{T}_N = \{0, 1, \ldots, N-1\}$, starting from an invariant state.

–Bernardin-Funaki-SS 2021

–The systems derived have trilinear Γ, which may suggest a 'mechanical' basis for 'trilinearity'.

–We are not aware however of derivations of bilinear, but not trilinear Γ coupled KPZ/SBE systems.

Remark about different speeds

For the system

$$\partial_t u^i = c_i \nabla u^i + \Delta u^i + \gamma^i_{j,k} \nabla \left(u^j u^k \right) + \nabla \xi^i_t$$

when all the $\{c_i\}$ are *different*,
several fluctuation classes are mentioned in the literature.

Popkov-Schadschneider-Schimidt-Schütz 2015

–In an 'ABC' exclusion model of two types of particles, e.g. $n = 2$, whose characteristic speeds are different, Cannizzaro-Gonçalves-Misturini-Occelli 2025 show the associated fluctuation field limits, in different reference frames, are independent.

Formally, one gets 'KPZ, KPZ'.

In the ‘same speeds’
or as it seems to be called ‘umbilical’ setting,
these drifts can be absorbed by Galilean shift in the equations.

–In BFS 2021, the characteristic speeds of the types were all the same.

–As mentioned, the fluctuation class picture in this setting is less resolved.

‘Umbilical’ class properties

Consider the coupled KPZ/SBE equations in the ‘same speeds’ or ‘umbilical’ setting.

–Define the ‘two-point’ function

$$S(x,t) = \left\{ E[u^i(x,t)u^j(0,0)] \right\}_{i,j}.$$

One would like to understand scalings such as

$$S(x,t) \approx t^{-\beta} G(xt^{-\beta}),$$

corresponding to the ‘lateral’ correlations.

The ‘transversal’ correlations would correspond to say looking at current across the origin: One might suspect

$$\vec{J}_0(t) = \vec{m}t + t^{\delta}\vec{\zeta}_t,$$

where $\vec{\zeta}_t$ is a random process.

–Presumably, there should be ‘universality’ of G and $\vec{\zeta}_t$ in the class identified by β and δ.

Case: $n = 1$

When there is only one equation,

$$\partial_t u = \frac{1}{2}\nabla u + \gamma\nabla u + \nabla\xi_t,$$

we remind on the progress in the last 10 years:

–When $\gamma \neq 0$, we have $\beta = 2/3$ and $\delta = 1/3$,
and universality properties of $G(z)$ are known and established.

–When $\gamma = 0$, the system is diffusive/Gaussian:
$\beta = 1/2$ and $\delta = 1/4$

Starting from an invariant measure, the lateral two-point scalings were found first for the ‘PNG’ model and shown for TASEP (Ferrari-Spohn 2006).

It was also shown for the KPZ/SBE model
(Borodin-Corwin-Ferrari-Veto 2015), all when $\gamma \neq 0$.

–Of course, all of these models are in the ‘KPZ’ class.

Case: $n \geq 2$

The system properties should be invariant under orthogonal transfomation $\sigma \in O(n)$.

–Consider $v = \sigma u$. The system under this map would read

$$\partial_t v^i = \frac{1}{2}\Delta v^i + (\sigma \circ \Gamma)^i_{j,k} \nabla \left(v^j v^k \right) + (\sigma \xi)^i_t$$

where

$$\begin{aligned}(\sigma \circ \Gamma)^i_{j,k} &= \sum_{i',j',k'} \sigma_{i,i'} \Gamma^{i'}_{j',k'} \sigma^{-1}_{j',j} \sigma^{-1}_{k',k} \\ &= \sum_{i',j',k'} \sigma_{i,i'} \Gamma^{i'}_{j',k'} \sigma_{j,j'} \sigma_{k,k'},\end{aligned}$$

since σ is orthogonal, e.g. $\sigma^{-1}_{j',j} = \sigma_{j,j'}$

–Note also that $\xi^v_t = \sigma \xi_t \overset{d}{=} \xi_t$.

Conjectured fluctuation classes

We briefly discuss possible 'umbilical' fluctuation classes.

–When $n = 2$, there are some numerical results and conjectures. We may represent a trilinear Γ as

$$\left(\gamma^1_{1,1} = a_3, \gamma^1_{1,2} = a_2, \gamma^1_{2,2} = a_1, \gamma^2_{2,2} = a_0\right)$$

–Roy-Dhar-Khanin-Kulkarni-Spohn 2024,
–Roy-Dhar-Kulkarni-Spohn 2025: Aside from the orbits of $(a_3, 0, 0, 0)$ and $(0, 0, 0, a_0)$, the scaling exponent $\beta = 3/2$ and $\delta = 1/3$ (or close to them) hold. Also, G and ζ_t may depend on the values of Γ.

–Schmidt-Krajnik-Popkov 2025: Also the exponents β, δ and G depend on Γ, and so a *continuum* of fluctuation classes may exist.

Decoupleability

We say that a Γ is 'fully decoupleable' if there is $\sigma \in O(n)$ where the only nonzero entries of $\sigma \circ \Gamma$ are $(\sigma \circ \Gamma)^i_{i,i}$.

It is 'partially decoupleable', if there is $\sigma \in O(n)$ where $(\sigma \circ \Gamma)^1_{i,j} = 0$ except when $i = j = 1$.

–This ties into the $n = 1$ picture.

If there is $\sigma \in O(n)$ so that $v = \sigma u$ 'fully decouples' into n 1D systems,

$$\partial_t v^i = \frac{1}{2}\Delta v + (\sigma \circ \Gamma)^i_{i,i}\left(\nabla v^i\right)^2 + \nabla \xi^{v,i}_t,$$

then of course $\beta = 3/2$, $\delta = 1/3$ (or diffusive scaling if $(\sigma\Gamma)^i_{i,i} = 0$), and G can be found.

Lemma:

When $n = 2$, and Γ is trilinear,
both notions are the same.

$$(\sigma \circ \Gamma)^1 = \begin{bmatrix} (\sigma \circ \Gamma)^1_{1,1} & 0 \\ 0 & 0 \end{bmatrix} \Rightarrow (\sigma \circ \Gamma)^2 = \begin{bmatrix} 0 & 0 \\ 0 & (\sigma \circ \Gamma)^2_{2,2} \end{bmatrix}.$$

On the other hand, when $n \geq 3$, there are partially decoupled Γ which are not fully decoupleable.

Case $n = 2$

When $n = 2$, although the orbit of $\Gamma = (a_3, a_2, a_1, a_0)$ is determined by these 4 parameters, one can identify the orbit with only 3 parameters:

Consider the rotation

$$\sigma_\theta = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$$

One may choose θ so that

$$(\sigma_\theta \circ \Gamma)^1_{2,2} = \mathcal{G}^1_{11} \cos\theta \sin^2\theta - \mathcal{G}^2_{22} \sin\theta \cos^2\theta$$
$$+ \mathcal{G}^2_{11}(-\sin^3\theta + 2\cos^2\theta \sin\theta) + \mathcal{G}^1_{22}(\cos^3\theta - 2\sin^2\theta\cos\theta) = 0.$$

Then, only three combinations suffice to identify the orbit.

However, orbits of fully decoupled tensors are determined by two parameters $\beta_1 = \gamma^1_{1,1}$ and $\beta_2 = \gamma^2_{2,2}$.

–Hence, one suspects a single relation to identify full decoupleability.

Lemma:

When $n = 2$,

Γ is fully (or partially) decoupled exactly when

$$a_3(a_1 - a_3) = a_2(a_2 - a_0).$$

There are a couple of ‘direct’ proofs, one by computing exactly when there is a σ_θ such that $\sigma_\theta \circ \Gamma$ is fully decoupled.

$$(\sigma_\theta \circ \Gamma)^1 = \begin{bmatrix} * & 0 \\ 0 & 0 \end{bmatrix}.$$

–Another, considers the differential equation

$$\partial_\theta \big(\sigma_\theta \circ \Gamma\big) = L\big(\sigma_\theta \circ \Gamma\big)$$

where the generator L is explicit, and solves for the decoupling condition.

–Note that $O(2) = SO(2) \oplus \{Id, \mathcal{N}\}$.
On the $O(2)$ orbit of a fully decoupled tensor Γ, corresponding to $\beta_1 \neq \beta_2 \neq 0$, there will be 8 reduced form tensors, w.r.t. $\pm\beta_1$, $\pm\beta_2$ and flips.

Cases $n \geq 3$

We found it difficult to solve directly for conditions when $n \geq 3$.

–The concept of ‘invariants’, going back to the 1800’s, will be useful.

Invariant theory approach.

An ‘invariant’ is a real polynomial function I of the entries in Γ which is constant over $O(n)$ action:

$$I(\sigma \circ \Gamma) = I(\Gamma).$$

Computing invariants

Let p be any real polynomial of the entries of Γ.
Then,

$$I(\Gamma) := \frac{1}{|\mathcal{G}|} \int_{\mathcal{G}} p\big(\sigma \circ \Gamma\big)\, d\sigma$$

is an $\mathcal{G}$ invariant.

Molien's formula:

$$\Phi(\lambda) = \frac{1}{|\mathcal{G}|} \int_{\mathcal{G}} \frac{1}{\det\big(Id - \lambda \mathcal{L}_\sigma\big)}\, d\sigma = \sum_{d \geq 0} c_d \lambda^d$$

is a generating function where c_d is the number of linearly independent polynomial invariants of degree d.

For instance, when $\mathcal{G} = SO(2)$, the action $\sigma_\theta \circ \Gamma$ may be found via the representation

$$\mathcal{L}_\sigma = \begin{pmatrix} \cos^3\theta & 3\sin\theta\cos^2\theta & 3\sin^2\theta\cos\theta & \sin^3\theta \\ -\sin\theta\cos^2\theta & \cos^3\theta - 2\sin^2\theta\cos\theta & 2\sin\theta\cos^2\theta - \sin^3\theta & \sin^2\theta\cos\theta \\ \sin^2\theta\cos\theta & \sin^3\theta - 2\sin\theta\cos^2\theta & \cos^3\theta - 2\sin^2\theta\cos\theta & \sin\theta\cos^2\theta \\ -\sin^3\theta & 3\sin^2\theta\cos\theta & -3\sin\theta\cos^2\theta & \cos^3\theta \end{pmatrix}$$

acting on $\Gamma = (a_3, a_2, a_1, a_0)^t$.

–The integral of $p(\sigma_\theta \circ \Gamma)$ is now over 0 to 2π.

–Molien's: For $SO(2)$, there are 4 invariants, two of degree 2, and two of degree 4 which span. For $O(2)$, there are 3 invariants, two of degree 2 and one of degree 4.

–However, when $n \geq 3$, we did not find it easy to compute more than just a few invariants.

Generalities

It is known that there is a finite set of invariants $\mathcal{I}$ which generate all polynomial invariants:

E.G. any invariant can be written as a polynomial in the members of $\mathcal{I}$.

–Further, such an $\mathcal{I}$ separates $O(n)$ orbits, that is the values of $\mathcal{I}$ on a tensor Γ determine the orbit of Γ.

–However, unless $n \leq 3$, explicit generating sets $\mathcal{I}$ are not known.

For $n = 2$, we mentioned some of the calculations.

For $n = 3$, it was a difficult problem, solved only recently by Olive-Auffray 2014.

Difficulties

In principle, given a basis $\mathcal{I}$, one may consider the values of $\mathcal{I}$ on a generic fully decoupled tensor in reduced form R, that is where only the diagonal entries $\Gamma^i_{ii} = \beta_i$ may not vanish.

Then, one may try to find relations between the expressions $\{I_1(R), I_2(R), \ldots, I_k(R)\}$ of $I_i \in \mathcal{I}$ on R by eliminating $\{\beta_i\}$.

These relations, like '$I_1(\Gamma) + 10 I_2^3(\Gamma) - 2 I_7(\Gamma) = 0$', would then be necessary conditions for full decoupleability of Γ.

However, in $n = 3$, even where $\mathcal{I}$ is explicitly given (Olive-Auffray 2014) with 13 members, we could not find these relations, even using Mathematica.

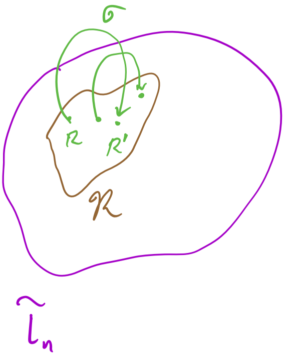
Alternative route

We now restrict to $n = 3$, more difficult than $n = 2$.

–The tack now taken is to look at invariants with respect to the *stabilizers* of fully decoupled or partially decoupled tensors in reduced forms $\mathcal{R}$.

–These are *smaller* sets.

–Subgroup $G_R \subset O(n)$ of maps $\sigma : \mathcal{R} \to \mathcal{R}$.



We have already mentioned the reduced form for fully decoupleable tensors:

$$R^1 = \begin{bmatrix} \beta_1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, R^2 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & \beta_2 & 0 \\ 0 & 0 & 0 \end{bmatrix}, R^3 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & \beta_3 \end{bmatrix}$$

–A reduced form R for partially decoupled tensors is

$$R^1 = \begin{bmatrix} a_3 & a_2 & 0 \\ a_2 & a_1 & 0 \\ 0 & 0 & 0 \end{bmatrix},\ R^2 = \begin{bmatrix} a_2 & a_1 & 0 \\ a_1 & a_0 & 0 \\ 0 & 0 & 0 \end{bmatrix},\ R^3 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & \beta_3 \end{bmatrix}$$

where to make a choice the third equation splits off from the others.

It turns out a generating set

$$\mathcal{J} = \{q_j\} : \mathcal{R} \to \mathcal{R}$$

of invariants with respect to the stabliizer group G_R on this smaller set $\mathcal{R}$ of tensors can be computed.

Necessity

Now all the invariants in $\mathcal{I}$ will be invariants with respect to the stabilizer group G_R on $\mathcal{R}$:

$$I_i(\sigma \circ R) = I_i(R) \text{ for } \sigma \in G_R \text{ and } R \in \mathcal{R}.$$

Hence, I_i can be written as polynomials of the generating set $\mathcal{J} = \{q_i\}$.

–Namely,

$$I_i(R) = F_i(\{q_j(R)\}).$$

Suppose $\mathcal{J}$ can be extended as $O(n)$ invariants on a *specified* $O(n)$-invariant subset $\mathcal{T}'$ of $\mathcal{T}_n$, which includes the orbit space of the reduced form tensors:

$$\mathcal{T}_n \supset \mathcal{T}' \supset \mathcal{S} = \cup_{R \in \mathcal{R}} \{O(n) \text{ orbit of } R\}.$$

–Let

$$\widetilde{\mathcal{J}} = \{\widetilde{q}_j\} : \mathcal{T}' \to \mathcal{T}'$$

denote the extension of $\mathcal{J}$.

We now 'lift' the previous specification to $\Gamma \in \mathcal{T}'$:

$$I_i(\Gamma) = F_i\big(\{\widetilde{q}_j(\Gamma)\}\big).$$

We claim belonging to $\mathcal{T}'$ and these relations are *necessary* for membership of $\Gamma \in \mathcal{S}$.

–Indeed, if $\Gamma \in \mathcal{S}$, it belongs to $\mathcal{T}'$. Let also $\hat{R} \in \mathcal{R}$ be a reduced form tensor on the $O(n)$ orbit of Γ.

–Then,

$$I_i(\Gamma) = I_i(\hat{R}) \;\text{ and }\; \widetilde{q}_j(\Gamma) = \widetilde{q}_j(\hat{R}) = q_j(\hat{R}).$$

–Since $I_i(\hat{R}) = F_i\big(\big\{q_j(\hat{R})\big\}\big)$, the above relations hold.

Sufficiency

It may be that the relations on $\mathcal{T}'$ specify a larger set of tensors than $\mathcal{S}$ ($=$ to the $O(n)$ orbits of $\mathcal{R}$).

–To guarantee that a tensor $\Gamma \in \mathcal{T}'$ belongs to $\mathcal{S}$, we impose the *solvability* condition:

There is $R \in \mathcal{R}$ where

$$\widetilde{q}_i(\Gamma) = \widetilde{q}_i(R).$$

–Indeed, if such R can be found, it plays the role of $\hat{R}$ in the previous slide, and the relations on $\mathcal{T}'$ are satisfied.

–This is a condition on a ‘small’ set of G_R-invariants $\{q_i\}$.

Decoupleability when $n = 3$

Ingredients include the following:

–There is a basis $\mathcal{I}$ consisting of 13 invariants,
Olive-Auffray 2014, in the study of 'elasiticity' of materials.

–To give it (for the record), we decompose

$$\Gamma = \frac{1}{n+2}\mathcal{D} + \mathcal{B}$$

into the sum of a 'trace-free' tensor $\mathcal{D}$ and a rank-one tensor $\mathcal{B}$ with the same trace vector $u = \left(\mathrm{Tr}\,\Gamma^1, \mathrm{Tr}\,\Gamma^2, \mathrm{Tr}\,\Gamma^2\right)$:

$$\mathcal{B}^i_{j,k} = \frac{1}{n+2}\left[u_i(e_j \cdot e_k) + u_j(e_i \cdot e_k) + u_k(e_i \cdot e_j)\right]$$

Let also

$$v_m = \mathcal{D}^i_{j,k}\,\mathcal{D}^i_{j,\ell}\,\mathcal{D}^k_{\ell,m}, \qquad w_m = \mathcal{D}^i_{j,m}\,u_i u_j$$

–Then, OA's basis is as follows:

$$\begin{array}{lll}
H_2 = \mathcal{D}^i_{j,k}\,\mathcal{D}^i_{j,k} & J_2 = u_i^2 & H_4 = \mathcal{D}^i_{j,k}\,\mathcal{D}^i_{j,\ell}\,\mathcal{D}^p_{q,k}\,\mathcal{D}^p_{q,\ell} \\
J_4 = \mathcal{D}^i_{j,k}\,u_k\,\mathcal{D}^\ell_{j,\ell}\,u_\ell & K_4 = \mathcal{D}^i_{j,k}\,\mathcal{D}^i_{j,\ell}\,\mathcal{D}^k_{\ell,p}\,u_p & L_4 = \mathcal{D}^i_{j,k}\,u_k u_j u_i \\
H_6 = v_i^2 & J_6 = \mathcal{D}^i_{j,k}\,\mathcal{D}^i_{j,\ell}\,u_k\,\mathcal{D}^\ell_{p,q}\,u_p u_q & K_6 = v_k w_k \\
L_6 = \mathcal{D}^i_{j,k}\,\mathcal{D}^i_{j,\ell}\,u_k v_\ell & M_6 = \mathcal{D}^i_{j,k}\,\mathcal{D}^p_{q,k}\,u_i u_j u_p u_q & H_8 = \mathcal{D}^i_{j,k}\,\mathcal{D}^i_{j,\ell}\,u_k\,\mathcal{D}^p_{q,\ell}\,\mathcal{D}^p_{q,r}\,v_r \\
H_{10} = \mathcal{D}^i_{j,k}\,v_i v_j v_k. & &
\end{array}$$

–Here, we use the Einstein summation convention.

Useful matrix

–It can be computed that the quadratic form of the 3×3 matrix

$$\Gamma^{*,2}_{k,\ell} = \sum_{i,j} \gamma^i_{j,k}\gamma^i_{j,\ell}$$

is invariant with respect to $O(3)$ action on Γ.

–For $R \in \mathcal{R}$, as before the set of fully decoupled tensors in reduced form, parametrized by $\beta_1, \beta_2, \beta_3$, note that

$$R^{*,2} = \begin{bmatrix} \beta_1^2 & 0 & 0 \\ 0 & \beta_2^2 & 0 \\ 0 & 0 & \beta_3^2 \end{bmatrix}.$$

The coefficients of the characteristic polynomial are the *symmetric functions*:

$$
\begin{aligned}
q_1(R) &= \beta_1^2\beta_2^2\beta_3^2, \\
q_2(R) &= \beta_1^2\beta_2^2 + \beta_1^2\beta_3^2 + \beta_2^2\beta_3^2, \\
q_3(R) &= \beta_1^2 + \beta_2^2 + \beta_3^2
\end{aligned}
$$

–These form a basis for the invariants with respect the stabilizer group $G_R = S_3$ action on $\mathcal{R}$.

–These extend to $\widetilde{q}_1(\Gamma), \widetilde{q}_2(\Gamma), \widetilde{q}_3(\Gamma)$ on full set of trilinear tensors $\mathcal{T}_n$ as the determinant, sum of minor determinants, and trace of $\Gamma^{*,2}$.

Solvability: One can <u>always</u> find an $\hat{R}$ so that

$$\widetilde{q}_i(\Gamma) = \widetilde{q}_i(\hat{R}),$$

by taking the diagonal entries $\{\beta_i\} = \{\sqrt{\widetilde{\beta}_i^2}\}$ where $\{\widetilde{\beta}_i\}$ are the eigenvalues of $\Gamma^{*,2}$.

–So, 'sufficiency' always holds.

Necessity: On $\mathcal{R}$, one can write $\mathcal{I}$ in terms of q_1, q_2, q_3, giving 13 relations.

–These extended to the full set of trilinear tensors $\mathcal{T}_n$ are then necessary.

Statement, BFSV 2025

When $n = 3$, we have 13 necessary and sufficient relations for $\Gamma \in \mathcal{T}_3$ to be fully decoupleable.

–These are the following (by Mathematica computation):

$$\begin{aligned}
H_2(\Gamma) &= 10\widetilde{q}_1 \\
J_2(\Gamma) &= \widetilde{q}_1 \\
J_4(\Gamma) &= 2\left(3\widetilde{q}_1^2 - 5\widetilde{q}_2\right) \\
J_6(\Gamma) &= 12\widetilde{q}_1^3 - 55\widetilde{q}_1\widetilde{q}_2 + 75\widetilde{q}_3 \\
L_6(\Gamma) &= 6\left(8\widetilde{q}_1^3 - 25\widetilde{q}_1\widetilde{q}_2 + 25\widetilde{q}_3\right) \\
H_6(\Gamma) &= 4\left(16\widetilde{q}_1^3 - 55\widetilde{q}_1\widetilde{q}_2 + 75\widetilde{q}_3\right) \\
H_8(\Gamma) &= 4\big(72\widetilde{q}_1^4 - 270\widetilde{q}_1^2\widetilde{q}_2 \\
&\quad +75\widetilde{q}_2^2 + 325\widetilde{q}_1\widetilde{q}_3\big) \\
H_{10}(\Gamma) &= 8\big(128\widetilde{q}_1^5 - 700\widetilde{q}_1^3\widetilde{q}_2 \\
&\quad +725\widetilde{q}_1\widetilde{q}_2^2 + 950\widetilde{q}_1^2\widetilde{q}_3 - 875\widetilde{q}_2\widetilde{q}_3\big)
\end{aligned}$$

$$\begin{aligned}
H_4(\Gamma) &= 2\left(22\widetilde{q}_1^2 - 15\widetilde{q}_2\right) \\
L_4(\Gamma) &= 2\left(\widetilde{q}_1^2 - 5\widetilde{q}_2\right) \\
K_4(\Gamma) &= 4\left(2\widetilde{q}_1^2 - 5\widetilde{q}_2\right) \\
K_6(\Gamma) &= 2\left(8\widetilde{q}_1^3 - 35\widetilde{q}_1\widetilde{q}_2 + 75\widetilde{q}_3\right) \\
M_6(\Gamma) &= 4\widetilde{q}_1^3 - 15\widetilde{q}_1\widetilde{q}_2 + 75\widetilde{q}_3
\end{aligned}$$

Partial decoupleability

We state the result for Γ to be partially decoupled, but not fully decoupled.

–The reason to disallow 'full decoupleability' is to lessen the number of parameters, and to avoid a piecewise description.

– Consider again partially decoupled tensors in form

$$R^1 = \begin{pmatrix} a_3 & a_2 & 0 \\ a_2 & a_1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, R^2 = \begin{pmatrix} a_2 & a_1 & 0 \\ a_1 & a_0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, R^3 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & \beta_3 \end{pmatrix}$$

–Let G be the associated $2 \times 2 \times 2$ subtensor.

A basis of invariants of the stabilizer subgroup with respect to $O(3)$ action can be computed:

$$\begin{aligned} q_1 &= \beta_3^2 \\ q_2 &= \|\mathrm{Tr}(G)\|^2 \\ q_3 &= \det(G^{*,2}) \\ q_4 &= \mathrm{Tr}(G^{*,2}). \end{aligned}$$

–A choice of 4 invariants in the Olive-Auffray basis,

$$H_2, H_4, J_2, L_4,$$

can be written in terms of these, for Γ restricted to these tensors.

Namely:

$$\begin{aligned} H_2 &= 10q_1 - 15q_2 + 25q_4 \\ H_4 &= 44q_1^2 - 42q_1q_2 + 144q_2^2 - 30q_3 \\ &\quad + 100q_1q_4 - 420q_2q_4 + 320q_4^2 \\ J_2 &= q_1 + q_2 \\ L_4 &= 2q_1^2 - 6q_1q_2 + 2q_2^2 - 10q_3 - \frac{5}{2}q_2q_4 + \frac{5}{2}q_4^2. \end{aligned}$$

Let us now invert these:

$$
\begin{aligned}
q_1 &= \frac{H_2^2 - 2H_4 - 3H_2J_2 + 6J_2^2 + 6L_4}{9(10J_2 - H_2)} \\
q_2 &= J_2 - \widetilde{q}_1 \\
q_3 &= \frac{1}{11250}\big(- 8H_2^2 + 25H_4 + 60H_2J_2 \\
&\qquad + 1500J_2^2 - 1200L_4 - 11250J_2\widetilde{q}_1 + 11250\widetilde{q}_1^2\big) \\
q_4 &= \frac{1}{25}\big(H_2 + 15J_2 - 25\widetilde{q}_1\big).
\end{aligned}
$$

–This suggests a way to extend $\{q_i\}$.

One may check the condition

$$10J_2 - H_2 = 0$$

is exactly the $n = 2$ condition for the $2 \times 2 \times 2$ subtensor G not to be decoupleable.

–The divisor is zero exactly when the partially decoupleable R is also fully decoupleable.

Let us define $\mathcal{R}$ to be the subset of these R's.

–Hence, the $\{q_i\}$ can be extended to $\{\widetilde{q}_i\}$ on the set

$$\mathcal{T}' = \big\{\Gamma : 10J_2(\Gamma) - H_2(\Gamma) \neq 0\big\},$$

which contains $\mathcal{S} = \cup_{R \in \mathcal{R}} \big\{O(n) \text{ orbits of } R\}$, the set of partially but not fully decoupleable tensors.

Necessity

For Γ to be partially but not fully decoupleable,

it is then necessary for $\Gamma \in \mathcal{T}'$

and for Γ to satisfy the relations of the Olive-Auffray basis $\mathcal{I}$ in terms of $\{q_i\}$ on $\mathcal{R}$, when extended to $\mathcal{T}'$.

–These relations computed by Mathematica are explicit but long, and so omitted!

Please see BFSV 2025.

Sufficiency/Solvability

Sufficiency is provided once we can solve

$$\widetilde{q}_i(\Gamma) = q_i(\hat{R})$$

for some $\hat{R} \in \mathcal{R}$.

–Since $\mathcal{R}$ is a ‘small’ set, we can explicitly solve for this condition.

For the record, ‘solvability’ for $\Gamma \in \widetilde{\mathcal{T}}$ holds exactly when

$$\widetilde{q}_1(\Gamma) \geq 0, \text{ and } \widetilde{q}_2(\Gamma) \geq 0,$$

and the following conditions
depending on whether $\widetilde{q}_2(\Gamma) > 0$ or $\widetilde{q}_2(\Gamma) = 0$ hold:

–When $\widetilde{q}_2(\Gamma) > 0$, we must have

$$-\widetilde{q}_2^4 + 4\widetilde{q}_2^2\widetilde{q}_3 - 16\widetilde{q}_3^2 + 2\widetilde{q}_2^3\widetilde{q}_4 - 8\widetilde{q}_2\widetilde{q}_3\widetilde{q}_4 \\ - 2\widetilde{q}_2^2\widetilde{q}_4^2 + 8\widetilde{q}_3\widetilde{q}_4^2 + 2\widetilde{q}_2\widetilde{q}_4^3 - \widetilde{q}_4^4 \geq 0.$$

–When $\widetilde{q}_2(\Gamma) = 0$, we must have

$$\widetilde{q}_3, \widetilde{q}_4 \geq 0 \text{ and } \widetilde{q}_3 = (\widetilde{q}_4)^2/4.$$

Discussion

1. If $R = \sigma \circ \Gamma$ is fully or partially decoupled, one can identify this 'decoupling' $\sigma \in O(n)$ in the 'generic' case, that is when $\Gamma^{*,2}$ has distinct eigenvalues.

–It turns out σ is one of 2^n transformations $\rho \in O(n)$ where $\rho\Gamma^{*,2}\rho^t = R^{*,2}$.

2. When $n = 3$, say for full decoupleability, can one do with a less number of necessary conditions? There may be 'syzygies/redundancies' in Olive-Auffray's basis.

–In a trilinear tensor $\Gamma \in O(3)$, there are 10 parameters.

–Since a rotation consists of an axis $\vec{n}$ (two parameters), and an angle (one parameter), it would reduce to 7 independent combinations.

–Since a fully decoupled tensor is identified with three parameters, this would suggest 4 relations.

3. In principle, one may find necessary/sufficient conditions also for $n \geq 4$ provided that a basis of $O(n)$ invariants can be found.

–Even the dimension of such a basis is not yet known.

4. Understanding bilinear, but not trilinear coupled KPZ/SBE systems is also of interest.

–Roy-Dhar-Kulkarni-Spohn 2025 suggest from numerical experiments that the transversal and lateral scalings are ‘near’ KPZ $(1/3, 2/3)$, but the structure functions depend on Γ.