Two-Sided Markets: Bulow-Klemperer-Style Results for Welfare

Yannai A. Gonczarowski

Harvard

SLMath (MSRI) Summer School
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Based upon (but all typos are my own):

Bulow-Klemperer-Style Results for Welfare Maximization in Two-Sided Markets, Moshe Babaioff, Kira Goldner, Y.A.G., 2020

Two Sided Markets ("Double Auctions")

- Each of m_S **sellers** holds one item. All items identical.
- Each of m_B (potential) **buyers** is interested in (any) one item.

- Each seller j has **private cost** $s_j \geq 0$ for parting with her item.
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- A **trade** is a specification of a set of sellers (to part with their items) and an equal-sized set of buyers (to obtain these items). **Efficient** if maximizes the **gains-from-trade**:

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- Ideal goal: a **mechanism** (function from all values and costs to a trade + payment/charge for each participant) that is:
  - **Individually rational (IR)** — allows voluntary participation.
  - **Incentive compatible (IC)** — incentivizes truthful reporting.
  - Weakly **budget balanced (BB)** — does not lose money ("IR for the auctioneer").
  - **Efficient** — output trade efficient w.r.t. input costs/values.

Myerson and Sattherthwaite's Seminal Impossibility

VCG is a (generally applicable) IR, IC, efficient mechanism.

- Output efficient trade.
- Charge each trading buyer her minimum trading bid.
- Pay each trading seller her maximum trading bid.

Example

For one buyer with value $b = 10$ and one seller with cost $s = 9$:

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Theorem (Myerson and Satterthwaite, 1983)

Even for one seller and one buyer ($m_S = m_B = 1$), there is no mechanism that is IR, IC, BB, and efficient.

The "Go To" Road to a Positive Result

- "First best" efficiency infeasible!
- "Go to" mechanism design approach: maintain **feasibility** constraints (IR, IC, BB), relax efficiency.
  - Assume values and costs are independently drawn from some distribution, find feasible mechanism with optimal expected efficiency ("second best").

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    - Needs to be carefully **tailored to the Bayesian prior**.
    - Known to be extremely **complex**, eludes precise description.
- $\Rightarrow$ As in many mechanism-design settings, tradeoff between efficiency on the one hand, and on the other hand both mechanism simplicity and amount of knowledge required by mechanism.

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- Optimal mechanism, even for i.i.d. buyers, requires some information about the distributions.
- Bulow-Klemperer (1996): with i.i.d. buyers, under assumptions on the distribution, if we can **recruit** one more **similar** buyer (=i.i.d. same distribution), we can "beat" the tradeoff from the last slide: $\exists$ a **simple**, **prior-independent**, **feasible** (IR & IC) mechanism that in the **augmented market** gives expected **revenue** $\geq$ optimal revenue in the original market.

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This lecture: Bulow-Klemperer-style results for two-sided markets.

- "Beat the tradeoff"! A simple, prior-independent, feasible (**IR**, **IC**, **BB**) mechanism that in an augmented market gives expected **efficiency** $\geq$ optimal efficiency in the original market.

Main Result

Setting:

- Market with m_S sellers, m_B buyers.
- Values and costs drawn i.i.d. from a distribution F.
- **Augmented market**: has one more buyer with value drawn independently from F. (m_S sellers, $m_B\mathbf{+1}$ buyers.)

Theorem (Main Result — Informal)

There exists mechanism that is a simple, prior-independent (=does not require any information about F), IR, IC, and BB, such that this mechanism in the augmented market has expected gains-from-trade at least as high as the optimal-yet-infeasible VCG mechanism in the original market.

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- Same result also if adding a seller rather than a buyer.
  - Aesthetic preference to add buyer: same pre-trade welfare.
  - Same will hold also for all other results we'll see today.

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70	20
60	45
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Theorem (Main Result — Formal Restatement)

$$\forall m_S, m_B, \forall F : \qquad \mathrm{BTR}(m_S, m_B+1) \geq \mathrm{OPT}(m_S, m_B).$$

Proof: $\mathrm{BTR}(m_S, m_B+1) \geq \mathrm{OPT}(m_S, m_B)$

- We will prove that

$$\mathrm{OPT}(m_S, m_B+1) - \mathrm{BTR}(m_S, m_B+1) \;\leq\; \mathrm{OPT}(m_S, m_B+1) - \mathrm{OPT}(m_S, m_B).$$

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  1. Draw m_S+m_B+1 values i.i.d. from F:
  $$x^{(1)} \geq \cdots\cdots \geq x^{(m_S)} \geq x^{(m_S+1)} \geq \cdots\cdots \geq x^{(m_S+m_B+1)}.$$
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- For any $x^{(1)} \geq \cdots \geq x^{(m_S+m_B+1)}$, we will prove in expectation over Step 2 that

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  $$\mathrm{OPT}(m_S, m_B{+}1) - \mathrm{BTR}(m_S, m_B{+}1) \;\leq\; \mathrm{OPT}(m_S, m_B{+}1) - \mathrm{OPT}(m_S, m_B).$$

- Couple the two markets:
  1. Draw $m_S{+}m_B{+}1$ values i.i.d. from F:

     $$\underbrace{x^{(1)} \geq \cdots\cdots \geq x^{(m_S)}}_{\text{trading buyers \& nontrading sellers}} \geq \underbrace{x^{(m_S+1)} \geq \cdots\cdots \geq x^{(m_S+m_B+1)}}_{\text{nontrading buyers \& trading sellers}}.$$

  2. Uniformly at random assign m_S as sellers, m_B as old buyers, 1 as new buyer.

- For any $x^{(1)} \geq \cdots \geq x^{(m_S+m_B+1)}$, we will prove in expectation over Step 2 that

  $$\mathbb{E}[\mathrm{OPT}_{aug}] - \mathbb{E}[\mathrm{BTR}_{aug}] \;\leq\; \mathbb{E}[\mathrm{OPT}_{aug}] - \mathbb{E}[\mathrm{OPT}_{orig}].$$

	$\mathbb{E}[\mathrm{OPT}_{aug}] - \mathbb{E}[\mathrm{BTR}_{aug}]$	$\mathbb{E}[\mathrm{OPT}_{aug}] - \mathbb{E}[\mathrm{OPT}_{orig}]$

Proof: $\mathrm{BTR}(m_S, m_B+1) \geq \mathrm{OPT}(m_S, m_B)$

- We will prove that

$$\mathrm{OPT}(m_S, m_B+1) - \mathrm{BTR}(m_S, m_B+1) \;\leq\; \mathrm{OPT}(m_S, m_B+1) - \mathrm{OPT}(m_S, m_B).$$

- Couple the two markets:
  1. Draw m_S+m_B+1 values i.i.d. from F:

$$\underbrace{x^{(1)} \geq \ldots\ldots \geq x^{(m_S)}}_{\text{trading buyers \& nontrading sellers}} \geq \underbrace{x^{(m_S+1)} \geq \ldots\ldots \geq x^{(m_S+m_B+1)}}_{\text{nontrading buyers \& trading sellers}}.$$

  2. Uniformly at random assign m_S as sellers, m_B as old buyers, 1 as new buyer.

- For any $x^{(1)} \geq \cdots \geq x^{(m_S+m_B+1)}$, we will prove in expectation over Step 2 that

$$\mathbb{E}[\mathrm{OPT}_{aug}] - \mathbb{E}[\mathrm{BTR}_{aug}] \;\leq\; \mathbb{E}[\mathrm{OPT}_{aug}] - \mathbb{E}[\mathrm{OPT}_{orig}].$$

	$\mathbb{E}[\mathrm{OPT}_{aug}] - \mathbb{E}[\mathrm{BTR}_{aug}]$	$\mathbb{E}[\mathrm{OPT}_{aug}] - \mathbb{E}[\mathrm{OPT}_{orig}]$
diff $\neq 0$ if ...		$x^{(1)} \ldots\ldots\, x^{(m_S)}\, x^{(m_S+1)} \ldots\ldots$

Proof: $\mathrm{BTR}(m_S, m_B+1) \geq \mathrm{OPT}(m_S, m_B)$

- We will prove that

$$\mathrm{OPT}(m_S, m_B+1) - \mathrm{BTR}(m_S, m_B+1) \;\leq\; \mathrm{OPT}(m_S, m_B+1) - \mathrm{OPT}(m_S, m_B).$$

- Couple the two markets:
  1. Draw $m_S + m_B + 1$ values i.i.d. from F:

$$\underbrace{x^{(1)} \geq \ldots\ldots \geq x^{(m_S)}}_{\text{trading buyers \& nontrading sellers}} \geq \underbrace{x^{(m_S+1)} \geq \ldots\ldots \geq x^{(m_S+m_B+1)}}_{\text{nontrading buyers \& trading sellers}}.$$

  2. Uniformly at random assign m_S as sellers, m_B as old buyers, 1 as new buyer.

- For any $x^{(1)} \geq \cdots \geq x^{(m_S+m_B+1)}$, we will prove in expectation over Step 2 that

$$\mathbb{E}[\mathrm{OPT}_{aug}] - \mathbb{E}[\mathrm{BTR}_{aug}] \;\leq\; \mathbb{E}[\mathrm{OPT}_{aug}] - \mathbb{E}[\mathrm{OPT}_{orig}].$$

	$\mathbb{E}[\mathrm{OPT}_{aug}] - \mathbb{E}[\mathrm{BTR}_{aug}]$	$\mathbb{E}[\mathrm{OPT}_{aug}] - \mathbb{E}[\mathrm{OPT}_{orig}]$
diff $\neq 0$ if ...		$\overbrace{x^{(1)} \ldots\ldots x^{(m_S)}}^{\text{new buyer in top } m_S}\; x^{(m_S+1)} \ldots\ldots$

Proof: $\mathrm{BTR}(m_S, m_B+1) \geq \mathrm{OPT}(m_S, m_B)$

- We will prove that

$$\mathrm{OPT}(m_S, m_B+1) - \mathrm{BTR}(m_S, m_B+1) \;\leq\; \mathrm{OPT}(m_S, m_B+1) - \mathrm{OPT}(m_S, m_B).$$

- Couple the two markets:
  1. Draw m_S+m_B+1 values i.i.d. from F:

$$\underbrace{x^{(1)} \geq \ldots\ldots \geq x^{(m_S)}}_{\text{trading buyers \& nontrading sellers}} \geq \underbrace{x^{(m_S+1)} \geq \ldots\ldots \geq x^{(m_S+m_B+1)}}_{\text{nontrading buyers \& trading sellers}}.$$

  2. Uniformly at random assign m_S as sellers, m_B as old buyers, 1 as new buyer.

- For any $x^{(1)} \geq \cdots \geq x^{(m_S+m_B+1)}$, we will prove in expectation over Step 2 that

$$\mathbb{E}[\mathrm{OPT}_{aug}] - \mathbb{E}[\mathrm{BTR}_{aug}] \;\leq\; \mathbb{E}[\mathrm{OPT}_{aug}] - \mathbb{E}[\mathrm{OPT}_{orig}].$$

	$\mathbb{E}[\mathrm{OPT}_{aug}] - \mathbb{E}[\mathrm{BTR}_{aug}]$	$\mathbb{E}[\mathrm{OPT}_{aug}] - \mathbb{E}[\mathrm{OPT}_{orig}]$
diff $\neq 0$ if ...		$\underbrace{x^{(1)}\ldots\ldots x^{(m_S)}}^{\text{new buyer in top } m_S}\ x^{(m_S+1)}\ldots\ldots$
$\Pr[\text{diff} \neq 0]$		$m_S/(m_S+m_B+1)$

Proof: $\mathrm{BTR}(m_S, m_B+1) \geq \mathrm{OPT}(m_S, m_B)$

- We will prove that
$$\mathrm{OPT}(m_S, m_B+1) - \mathrm{BTR}(m_S, m_B+1) \;\leq\; \mathrm{OPT}(m_S, m_B+1) - \mathrm{OPT}(m_S, m_B).$$
- Couple the two markets:
  1. Draw $m_S + m_B + 1$ values i.i.d. from F:
$$\underbrace{x^{(1)} \geq \cdots\cdots \geq x^{(m_S)}}_{\text{trading buyers \& nontrading sellers}} \geq \underbrace{x^{(m_S+1)} \geq \cdots\cdots \geq x^{(m_S+m_B+1)}}_{\text{nontrading buyers \& trading sellers}}.$$
  2. Uniformly at random assign m_S as sellers, m_B as old buyers, 1 as new buyer.
- For any $x^{(1)} \geq \cdots \geq x^{(m_S+m_B+1)}$, we will prove in expectation over Step 2 that
$$\mathbb{E}[\mathrm{OPT}_{aug}] - \mathbb{E}[\mathrm{BTR}_{aug}] \;\leq\; \mathbb{E}[\mathrm{OPT}_{aug}] - \mathbb{E}[\mathrm{OPT}_{orig}].$$

	$\mathbb{E}[\mathrm{OPT}_{aug}] - \mathbb{E}[\mathrm{BTR}_{aug}]$	$\mathbb{E}[\mathrm{OPT}_{aug}] - \mathbb{E}[\mathrm{OPT}_{orig}]$
diff $\neq 0$ if ...		$\underbrace{x^{(1)} \cdots\cdots x^{(m_S)}}^{\text{new buyer in top } m_S}\ x^{(m_S+1)} \cdots\cdots$
$\Pr[\text{diff} \neq 0]$		$m_S/(m_S + m_B + 1)$
$\mathbb{E}[\mathrm{OPT}_{aug}]$	$x^{(1)} \cdots\cdots x^{(m_S)}\ x^{(m_S+1)} \cdots\cdots x^{(m_S+m_B)}$	

Proof: $\mathrm{BTR}(m_S, m_B+1) \geq \mathrm{OPT}(m_S, m_B)$

- We will prove that
  $$\mathrm{OPT}(m_S, m_B+1) - \mathrm{BTR}(m_S, m_B+1) \leq \mathrm{OPT}(m_S, m_B+1) - \mathrm{OPT}(m_S, m_B).$$
- Couple the two markets:
  1. Draw m_S+m_B+1 values i.i.d. from F:
  $$\underbrace{x^{(1)} \geq \ldots\ldots \geq x^{(m_S)}}_{\text{trading buyers \& nontrading sellers}} \geq \underbrace{x^{(m_S+1)} \geq \ldots\ldots \geq x^{(m_S+m_B+1)}}_{\text{nontrading buyers \& trading sellers}}.$$
  2. Uniformly at random assign m_S as sellers, m_B as old buyers, 1 as new buyer.
- For any $x^{(1)} \geq \cdots \geq x^{(m_S+m_B+1)}$, we will prove in expectation over Step 2 that
  $$\mathbb{E}[\mathrm{OPT}_{aug}] - \mathbb{E}[\mathrm{BTR}_{aug}] \leq \mathbb{E}[\mathrm{OPT}_{aug}] - \mathbb{E}[\mathrm{OPT}_{orig}].$$

	$\mathbb{E}[\mathrm{OPT}_{aug}] - \mathbb{E}[\mathrm{BTR}_{aug}]$	$\mathbb{E}[\mathrm{OPT}_{aug}] - \mathbb{E}[\mathrm{OPT}_{orig}]$
diff $\neq 0$ if ...		$\overbrace{x^{(1)} \ldots\ldots x^{(m_S)}}^{\text{new buyer in top } m_S} x^{(m_S+1)} \ldots\ldots$
$\Pr[\text{diff} \neq 0]$		$m_S/(m_S+m_B+1)$
$\mathbb{E}[\mathrm{OPT}_{aug}]$	$\boxed{x^{(1)} \ldots\ldots x^{(m_S)}}\, x^{(m_S+1)} \ldots\ldots x^{(m_S+m_B)}$	

Proof: $\mathrm{BTR}(m_S, m_B+1) \geq \mathrm{OPT}(m_S, m_B)$

- We will prove that
$$\mathrm{OPT}(m_S, m_B+1) - \mathrm{BTR}(m_S, m_B+1) \;\leq\; \mathrm{OPT}(m_S, m_B+1) - \mathrm{OPT}(m_S, m_B).$$
- Couple the two markets:
  1. Draw $m_S + m_B + 1$ values i.i.d. from F:
$$\underbrace{x^{(1)} \geq \cdots\cdots \geq x^{(m_S)}}_{\text{trading buyers \& nontrading sellers}} \geq \underbrace{x^{(m_S+1)} \geq \cdots\cdots \geq x^{(m_S+m_B+1)}}_{\text{nontrading buyers \& trading sellers}}.$$
  2. Uniformly at random assign m_S as sellers, m_B as old buyers, 1 as new buyer.
- For any $x^{(1)} \geq \cdots \geq x^{(m_S+m_B+1)}$, we will prove in expectation over Step 2 that
$$\mathbb{E}[\mathrm{OPT}_{aug}] - \mathbb{E}[\mathrm{BTR}_{aug}] \;\leq\; \mathbb{E}[\mathrm{OPT}_{aug}] - \mathbb{E}[\mathrm{OPT}_{orig}].$$

	$\mathbb{E}[\mathrm{OPT}_{aug}] - \mathbb{E}[\mathrm{BTR}_{aug}]$	$\mathbb{E}[\mathrm{OPT}_{aug}] - \mathbb{E}[\mathrm{OPT}_{orig}]$
diff $\neq 0$ if ...		$\overbrace{x^{(1)} \cdots\cdots x^{(m_S)}}^{\text{new buyer in top } m_S} x^{(m_S+1)} \cdots\cdots$
$\Pr[\text{diff} \neq 0]$		$m_S/(m_S + m_B + 1)$
$\mathbb{E}[\mathrm{OPT}_{aug}]$	$\boxed{x^{(1)} \cdots\cdots x^{(m_S)}}\, x^{(m_S+1)} \cdots\cdots x^{(m_S+m_B)}$	
minus ...		$x^{(1)} \cdots \overset{\substack{\text{new buyer}\\ \downarrow}}{x^{(\nu)}} \cdots x^{(m_S)}\, x^{(m_S+1)} \cdots\cdots$

Proof: $\mathrm{BTR}(m_S, m_B+1) \geq \mathrm{OPT}(m_S, m_B)$

- We will prove that

$$\mathrm{OPT}(m_S, m_B+1) - \mathrm{BTR}(m_S, m_B+1) \;\leq\; \mathrm{OPT}(m_S, m_B+1) - \mathrm{OPT}(m_S, m_B).$$

- Couple the two markets:
  1. Draw $m_S + m_B + 1$ values i.i.d. from F:

$$\underbrace{x^{(1)} \geq \cdots\cdots \geq x^{(m_S)}}_{\text{trading buyers \& nontrading sellers}} \geq \underbrace{x^{(m_S+1)} \geq \cdots\cdots \geq x^{(m_S+m_B+1)}}_{\text{nontrading buyers \& trading sellers}}.$$

  2. Uniformly at random assign m_S as sellers, m_B as old buyers, 1 as new buyer.

- For any $x^{(1)} \geq \cdots \geq x^{(m_S+m_B+1)}$, we will prove in expectation over Step 2 that

$$\mathbb{E}[\mathrm{OPT}_{aug}] - \mathbb{E}[\mathrm{BTR}_{aug}] \;\leq\; \mathbb{E}[\mathrm{OPT}_{aug}] - \mathbb{E}[\mathrm{OPT}_{orig}].$$

	$\mathbb{E}[\mathrm{OPT}_{aug}] - \mathbb{E}[\mathrm{BTR}_{aug}]$	$\mathbb{E}[\mathrm{OPT}_{aug}] - \mathbb{E}[\mathrm{OPT}_{orig}]$
diff $\neq 0$ if …		$\overbrace{x^{(1)} \cdots\cdots x^{(m_S)}}^{\text{new buyer in top } m_S}\ x^{(m_S+1)} \cdots\cdots$
$\Pr[\text{diff} \neq 0]$		$m_S/(m_S + m_B + 1)$
$\mathbb{E}[\mathrm{OPT}_{aug}]$	$\boxed{x^{(1)} \cdots\cdots x^{(m_S)}}\ x^{(m_S+1)} \cdots\cdots x^{(m_S+m_B)}$	
minus …		$\boxed{x^{(1)} \cdots}\ \underset{\uparrow}{x^{(\nu)}}\ \boxed{\cdots\ x^{(m_S)}\ x^{(m_S+1)}} \cdots\cdots$ new buyer

Proof: $\mathrm{BTR}(m_S, m_B+1) \geq \mathrm{OPT}(m_S, m_B)$

- We will prove that
$$\mathrm{OPT}(m_S, m_B+1) - \mathrm{BTR}(m_S, m_B+1) \;\leq\; \mathrm{OPT}(m_S, m_B+1) - \mathrm{OPT}(m_S, m_B).$$
- Couple the two markets:
  1. Draw m_S+m_B+1 values i.i.d. from F:
$$\underbrace{x^{(1)} \geq \cdots\cdots \geq x^{(m_S)}}_{\text{trading buyers \& nontrading sellers}} \geq \underbrace{x^{(m_S+1)} \geq \cdots\cdots \geq x^{(m_S+m_B+1)}}_{\text{nontrading buyers \& trading sellers}}.$$
  2. Uniformly at random assign m_S as sellers, m_B as old buyers, 1 as new buyer.
- For any $x^{(1)} \geq \cdots \geq x^{(m_S+m_B+1)}$, we will prove in expectation over Step 2 that
$$\mathbb{E}[\mathrm{OPT}_{aug}] - \mathbb{E}[\mathrm{BTR}_{aug}] \;\leq\; \mathbb{E}[\mathrm{OPT}_{aug}] - \mathbb{E}[\mathrm{OPT}_{orig}].$$

	$\mathbb{E}[\mathrm{OPT}_{aug}] - \mathbb{E}[\mathrm{BTR}_{aug}]$	$\mathbb{E}[\mathrm{OPT}_{aug}] - \mathbb{E}[\mathrm{OPT}_{orig}]$
diff $\neq 0$ if …	$x^{(1)}\cdots\cdots x^{(m_S)}\ x^{(m_S+1)}\cdots\cdots$	$\underbrace{x^{(1)}\cdots\cdots x^{(m_S)}}_{\text{new buyer in top } m_S}\ x^{(m_S+1)}\cdots\cdots$
$\Pr[\text{diff} \neq 0]$		$m_S/(m_S+m_B+1)$
$\mathbb{E}[\mathrm{OPT}_{aug}]$	$\boxed{x^{(1)}\cdots\cdots x^{(m_S)}}\ x^{(m_S+1)}\cdots\cdots x^{(m_S+m_B)}$	
minus …		new buyer $\downarrow$ $\boxed{x^{(1)}\cdots}\ x^{(\nu)}\ \boxed{\cdots x^{(m_S)}\ x^{(m_S+1)}}\cdots\cdots$

Proof: $\mathrm{BTR}(m_S, m_B+1) \geq \mathrm{OPT}(m_S, m_B)$

- We will prove that

$$\mathrm{OPT}(m_S, m_B+1) - \mathrm{BTR}(m_S, m_B+1) \;\leq\; \mathrm{OPT}(m_S, m_B+1) - \mathrm{OPT}(m_S, m_B).$$

- Couple the two markets:
  1. Draw m_S+m_B+1 values i.i.d. from F:

$$\underbrace{x^{(1)} \geq \cdots\cdots \geq x^{(m_S)}}_{\text{trading buyers \& nontrading sellers}} \geq \underbrace{x^{(m_S+1)} \geq \cdots\cdots \geq x^{(m_S+m_B+1)}}_{\text{nontrading buyers \& trading sellers}}.$$

  2. Uniformly at random assign m_S as sellers, m_B as old buyers, 1 as new buyer.

- For any $x^{(1)} \geq \cdots \geq x^{(m_S+m_B+1)}$, we will prove in expectation over Step 2 that

$$\mathbb{E}[\mathrm{OPT}_{aug}] - \mathbb{E}[\mathrm{BTR}_{aug}] \;\leq\; \mathbb{E}[\mathrm{OPT}_{aug}] - \mathbb{E}[\mathrm{OPT}_{orig}].$$

	$\mathbb{E}[\mathrm{OPT}_{aug}] - \mathbb{E}[\mathrm{BTR}_{aug}]$	$\mathbb{E}[\mathrm{OPT}_{aug}] - \mathbb{E}[\mathrm{OPT}_{orig}]$
diff $\neq 0$ if ...	$x^{(1)}\cdots\cdots x^{(m_S)}\, \overbrace{x^{(m_S+1)}\cdots\cdots}^{\text{a seller}}$	$\overbrace{x^{(1)}\cdots\cdots x^{(m_S)}}^{\text{new buyer in top } m_S}\, x^{(m_S+1)}\cdots\cdots$
$\Pr[\text{diff} \neq 0]$		$m_S/(m_S+m_B+1)$
$\mathbb{E}[\mathrm{OPT}_{aug}]$	$\boxed{x^{(1)}\cdots\cdots x^{(m_S)}}\, x^{(m_S+1)}\cdots\cdots x^{(m_S+m_B)}$	
minus ...		new buyer $\downarrow$ $\boxed{x^{(1)}\cdots}\, x^{(\nu)}\, \boxed{\cdots x^{(m_S)}\, x^{(m_S+1)}}\cdots\cdots$

Proof: $\mathrm{BTR}(m_S, m_B+1) \geq \mathrm{OPT}(m_S, m_B)$

- We will prove that

$$\mathrm{OPT}(m_S, m_B+1) - \mathrm{BTR}(m_S, m_B+1) \;\leq\; \mathrm{OPT}(m_S, m_B+1) - \mathrm{OPT}(m_S, m_B).$$

- Couple the two markets:
  1. Draw m_S+m_B+1 values i.i.d. from F:

$$\underbrace{x^{(1)} \geq \cdots\cdots \geq x^{(m_S)}}_{\text{trading buyers \& nontrading sellers}} \geq \underbrace{x^{(m_S+1)} \geq \cdots\cdots \geq x^{(m_S+m_B+1)}}_{\text{nontrading buyers \& trading sellers}}.$$

  2. Uniformly at random assign m_S as sellers, m_B as old buyers, 1 as new buyer.
- For any $x^{(1)} \geq \cdots \geq x^{(m_S+m_B+1)}$, we will prove in expectation over Step 2 that

$$\mathbb{E}[\mathrm{OPT}_{aug}] - \mathbb{E}[\mathrm{BTR}_{aug}] \;\leq\; \mathbb{E}[\mathrm{OPT}_{aug}] - \mathbb{E}[\mathrm{OPT}_{orig}].$$

	$\mathbb{E}[\mathrm{OPT}_{aug}] - \mathbb{E}[\mathrm{BTR}_{aug}]$	$\mathbb{E}[\mathrm{OPT}_{aug}] - \mathbb{E}[\mathrm{OPT}_{orig}]$
diff $\neq 0$ if ...	$x^{(1)}\cdots\cdots x^{(m_S)}\, \overbrace{x^{(m_S+1)}}^{\text{a seller (seller } q)}\cdots\cdots$	$\overbrace{x^{(1)}\cdots\cdots x^{(m_S)}}^{\text{new buyer in top } m_S}\, x^{(m_S+1)}\cdots\cdots$
$\Pr[\text{diff} \neq 0]$		$m_S/(m_S + m_B + 1)$
$\mathbb{E}[\mathrm{OPT}_{aug}]$	$\boxed{x^{(1)}\cdots\cdots x^{(m_S)}}\, x^{(m_S+1)}\cdots\cdots x^{(m_S+m_B)}$	
minus ...		$\boxed{x^{(1)}\cdots}\, \underset{\text{new buyer}}{x^{(\nu)}}\, \boxed{\cdots x^{(m_S)}\; x^{(m_S+1)}}\cdots\cdots$

Proof: $\mathrm{BTR}(m_S, m_B{+}1) \geq \mathrm{OPT}(m_S, m_B)$

- We will prove that
$$\mathrm{OPT}(m_S, m_B{+}1) - \mathrm{BTR}(m_S, m_B{+}1) \leq \mathrm{OPT}(m_S, m_B{+}1) - \mathrm{OPT}(m_S, m_B).$$
- Couple the two markets:
  1. Draw $m_S{+}m_B{+}1$ values i.i.d. from F:
$$\underbrace{x^{(1)} \geq \ldots\ldots \geq x^{(m_S)}}_{\text{trading buyers \& nontrading sellers}} \geq \underbrace{x^{(m_S+1)} \geq \ldots\ldots \geq x^{(m_S+m_B+1)}}_{\text{nontrading buyers \& trading sellers}}.$$
  2. Uniformly at random assign m_S as sellers, m_B as old buyers, 1 as new buyer.
- For any $x^{(1)} \geq \cdots \geq x^{(m_S+m_B+1)}$, we will prove in expectation over Step 2 that
$$\mathbb{E}[\mathrm{OPT}_{aug}] - \mathbb{E}[\mathrm{BTR}_{aug}] \leq \mathbb{E}[\mathrm{OPT}_{aug}] - \mathbb{E}[\mathrm{OPT}_{orig}].$$

	$\mathbb{E}[\mathrm{OPT}_{aug}] - \mathbb{E}[\mathrm{BTR}_{aug}]$	$\mathbb{E}[\mathrm{OPT}_{aug}] - \mathbb{E}[\mathrm{OPT}_{orig}]$
diff $\neq 0$ if …	$x^{(1)}\ldots\ldots x^{(m_S)}\, \overbrace{x^{(m_S+1)}}^{\text{a seller (seller } q)}\ldots\ldots$	$\overbrace{x^{(1)}\ldots\ldots x^{(m_S)}}^{\text{new buyer in top } m_S}\, x^{(m_S+1)}\ldots\ldots$
$\Pr[\text{diff} \neq 0]$	$m_S/(m_S + m_B + 1)$	$m_S/(m_S + m_B + 1)$
$\mathbb{E}[\mathrm{OPT}_{aug}]$	$\boxed{x^{(1)}\ldots\ldots x^{(m_S)}}\, x^{(m_S+1)}\ldots\ldots x^{(m_S+m_B)}$	
minus …		$\boxed{x^{(1)}\ldots}\, \overset{\text{new buyer}}{\overset{\downarrow}{x^{(\nu)}}}\, \boxed{\ldots x^{(m_S)}\, x^{(m_S+1)}}\ldots\ldots$

Model & Background

Market Augmentation

Proof

Further Research

Proof: $\mathrm{BTR}(m_S, m_B+1) \geq \mathrm{OPT}(m_S, m_B)$

- We will prove that

$$\mathrm{OPT}(m_S, m_B+1) - \mathrm{BTR}(m_S, m_B+1) \;\leq\; \mathrm{OPT}(m_S, m_B+1) - \mathrm{OPT}(m_S, m_B).$$

- Couple the two markets:
  1. Draw m_S+m_B+1 values i.i.d. from F:

$$\underbrace{x^{(1)} \geq \cdots\cdots \geq x^{(m_S)}}_{\text{trading buyers \& nontrading sellers}} \geq \underbrace{x^{(m_S+1)} \geq \cdots\cdots \geq x^{(m_S+m_B+1)}}_{\text{nontrading buyers \& trading sellers}}.$$

  2. Uniformly at random assign m_S as sellers, m_B as old buyers, 1 as new buyer.
- For any $x^{(1)} \geq \cdots \geq x^{(m_S+m_B+1)}$, we will prove in expectation over Step 2 that

$$\mathbb{E}[\mathrm{OPT}_{aug}] - \mathbb{E}[\mathrm{BTR}_{aug}] \;\leq\; \mathbb{E}[\mathrm{OPT}_{aug}] - \mathbb{E}[\mathrm{OPT}_{orig}].$$

	$\mathbb{E}[\mathrm{OPT}_{aug}] - \mathbb{E}[\mathrm{BTR}_{aug}]$		$\mathbb{E}[\mathrm{OPT}_{aug}] - \mathbb{E}[\mathrm{OPT}_{orig}]$
diff $\neq 0$ if …	$x^{(1)}\cdots\cdots x^{(m_S)}\,\overbrace{x^{(m_S+1)}}^{\text{a seller (seller } q)}\cdots\cdots$		$\overbrace{x^{(1)}\cdots\cdots x^{(m_S)}}^{\text{new buyer in top } m_S}\,x^{(m_S+1)}\cdots\cdots$
$\Pr[\text{diff} \neq 0]$	$m_S/(m_S+m_B+1)$	$=$	$m_S/(m_S+m_B+1)$
$\mathbb{E}[\mathrm{OPT}_{aug}]$	$\boxed{x^{(1)}\cdots\cdots x^{(m_S)}}\,x^{(m_S+1)}\cdots\cdots x^{(m_S+m_B)}$		
minus …			$\boxed{x^{(1)}\cdots}\,\underset{\text{new buyer}}{x^{(\nu)}}\,\boxed{\cdots x^{(m_S)}\,x^{(m_S+1)}}\cdots\cdots$

Proof: $\mathrm{BTR}(m_S, m_B+1) \geq \mathrm{OPT}(m_S, m_B)$

- We will prove that
$$\mathrm{OPT}(m_S, m_B+1) - \mathrm{BTR}(m_S, m_B+1) \;\leq\; \mathrm{OPT}(m_S, m_B+1) - \mathrm{OPT}(m_S, m_B).$$
- Couple the two markets:
  1. Draw m_S+m_B+1 values i.i.d. from F:
$$\underbrace{x^{(1)} \geq \ldots\ldots \geq x^{(m_S)}}_{\text{trading buyers \& nontrading sellers}} \geq \underbrace{x^{(m_S+1)} \geq \ldots\ldots \geq x^{(m_S+m_B+1)}}_{\text{nontrading buyers \& trading sellers}}.$$
  2. Uniformly at random assign m_S as sellers, m_B as old buyers, 1 as new buyer.
- For any $x^{(1)} \geq \cdots \geq x^{(m_S+m_B+1)}$, we will prove in expectation over Step 2 that
$$\mathbb{E}[\mathrm{OPT}_{aug}] - \mathbb{E}[\mathrm{BTR}_{aug}] \;\leq\; \mathbb{E}[\mathrm{OPT}_{aug}] - \mathbb{E}[\mathrm{OPT}_{orig}].$$

	$\mathbb{E}[\mathrm{OPT}_{aug}] - \mathbb{E}[\mathrm{BTR}_{aug}]$	$\mathbb{E}[\mathrm{OPT}_{aug}] - \mathbb{E}[\mathrm{OPT}_{orig}]$
diff $\neq 0$ if …	$x^{(1)}\ldots\ldots x^{(m_S)}\ \underbrace{x^{(m_S+1)}}^{\text{a seller (seller } q)}\ldots\ldots$	$\overbrace{x^{(1)}\ldots\ldots x^{(m_S)}}^{\text{new buyer in top } m_S}\ x^{(m_S+1)}\ldots\ldots$
$\Pr[\text{diff} \neq 0]$	$m_S/(m_S+m_B+1)$ =	$m_S/(m_S+m_B+1)$
$\mathbb{E}[\mathrm{OPT}_{aug}]$	$\boxed{x^{(1)}\ldots\ldots x^{(m_S)}}\ x^{(m_S+1)}\ldots\ldots x^{(m_S+m_B)}$	
minus …	$x^{(1)}\ldots\ \overset{\text{buyer } q}{\downarrow}\ x^{(\alpha)}\ldots x^{(m_S)}\ x^{(m_S+1)}\ldots\ldots$	$\boxed{x^{(1)}\ldots}\ \overset{\text{new buyer}}{\downarrow}\ x^{(\nu)}\ \boxed{\ldots x^{(m_S)}\ x^{(m_S+1)}}\ldots\ldots$

Proof: $\mathrm{BTR}(m_S, m_B+1) \geq \mathrm{OPT}(m_S, m_B)$

- We will prove that

$$\mathrm{OPT}(m_S, m_B+1) - \mathrm{BTR}(m_S, m_B+1) \;\leq\; \mathrm{OPT}(m_S, m_B+1) - \mathrm{OPT}(m_S, m_B).$$

- Couple the two markets:
  1. Draw m_S+m_B+1 values i.i.d. from F:

$$\underbrace{x^{(1)} \geq \cdots\cdots \geq x^{(m_S)}}_{\text{trading buyers \& nontrading sellers}} \geq \underbrace{x^{(m_S+1)} \geq \cdots\cdots \geq x^{(m_S+m_B+1)}}_{\text{nontrading buyers \& trading sellers}}.$$

  2. Uniformly at random assign m_S as sellers, m_B as old buyers, 1 as new buyer.

- For any $x^{(1)} \geq \cdots \geq x^{(m_S+m_B+1)}$, we will prove in expectation over Step 2 that

$$\mathbb{E}[\mathrm{OPT}_{aug}] - \mathbb{E}[\mathrm{BTR}_{aug}] \;\leq\; \mathbb{E}[\mathrm{OPT}_{aug}] - \mathbb{E}[\mathrm{OPT}_{orig}].$$

	$\mathbb{E}[\mathrm{OPT}_{aug}] - \mathbb{E}[\mathrm{BTR}_{aug}]$		$\mathbb{E}[\mathrm{OPT}_{aug}] - \mathbb{E}[\mathrm{OPT}_{orig}]$
diff $\neq 0$ if ...	$x^{(1)}\cdots\cdots x^{(m_S)} \overbrace{x^{(m_S+1)}}^{\text{a seller (seller } q)}\cdots\cdots$		$\overbrace{x^{(1)}\cdots\cdots x^{(m_S)}}^{\text{new buyer in top } m_S} x^{(m_S+1)}\cdots\cdots$
$\Pr[\text{diff} \neq 0]$	$m_S/(m_S+m_B+1)$	$=$	$m_S/(m_S+m_B+1)$
$\mathbb{E}[\mathrm{OPT}_{aug}]$	$\boxed{x^{(1)}\cdots\cdots x^{(m_S)}}\, x^{(m_S+1)}\cdots\cdots x^{(m_S+m_B)}$		
minus ...	$\boxed{x^{(1)}\cdots\cdot}\, \overset{\text{buyer } q \downarrow}{x^{(\alpha)}} \boxed{\cdot\cdot\, x^{(m_S)}\, x^{(m_S+1)}}\cdots\cdots$		$\boxed{x^{(1)}\cdots}\, \overset{\text{new buyer} \downarrow}{x^{(\nu)}} \boxed{\cdots x^{(m_S)}\, x^{(m_S+1)}}\cdots\cdots$

Proof: $\quad \mathrm{BTR}(m_S, m_B+1) \geq \mathrm{OPT}(m_S, m_B)$

- We will prove that
$$\mathrm{OPT}(m_S, m_B+1) - \mathrm{BTR}(m_S, m_B+1) \;\leq\; \mathrm{OPT}(m_S, m_B+1) - \mathrm{OPT}(m_S, m_B).$$
- Couple the two markets:
  1. Draw m_S+m_B+1 values i.i.d. from F:
$$\underbrace{x^{(1)} \geq \cdots\cdots \geq x^{(m_S)}}_{\text{trading buyers \& nontrading sellers}} \geq \underbrace{x^{(m_S+1)} \geq \cdots\cdots \geq x^{(m_S+m_B+1)}}_{\text{nontrading buyers \& trading sellers}}.$$
  2. Uniformly at random assign m_S as sellers, m_B as old buyers, 1 as new buyer.
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$$\mathbb{E}[\mathrm{OPT}_{aug}] - \mathbb{E}[\mathrm{BTR}_{aug}] \;\leq\; \mathbb{E}[\mathrm{OPT}_{aug}] - \mathbb{E}[\mathrm{OPT}_{orig}].$$

	$\mathbb{E}[\mathrm{OPT}_{aug}] - \mathbb{E}[\mathrm{BTR}_{aug}]$	$\mathbb{E}[\mathrm{OPT}_{aug}] - \mathbb{E}[\mathrm{OPT}_{orig}]$
diff $\neq 0$ if ...	$x^{(1)}\ldots\ldots x^{(m_S)}\ \overbrace{x^{(m_S+1)}}^{\text{a seller (seller } q)}\ldots\ldots$	$\overbrace{x^{(1)}\ldots\ldots x^{(m_S)}}^{\text{new buyer in top } m_S}\ x^{(m_S+1)}\ldots\ldots$
$\Pr[\text{diff} \neq 0]$	$m_S/(m_S+m_B+1)$ =	$m_S/(m_S+m_B+1)$
$\mathbb{E}[\mathrm{OPT}_{aug}]$	$\boxed{x^{(1)}\ldots\ldots x^{(m_S)}}\ x^{(m_S+1)}\ldots\ldots x^{(m_S+m_B)}$	
minus ...	$\boxed{x^{(1)}\ldots}\ \overset{\text{buyer } q}{\overset{\downarrow}{x^{(\alpha)}}}\ \boxed{\ldots x^{(m_S)}\ x^{(m_S+1)}}\ldots\ldots$	$\boxed{x^{(1)}\ldots}\ \overset{\text{new buyer}}{\overset{\downarrow}{x^{(\nu)}}}\ \boxed{\ldots x^{(m_S)}\ x^{(m_S+1)}}\ldots\ldots$
diff$+x^{(m_S+1)}$		$\text{val} \sim U\{x^{(1)}, \ldots, x^{(m_S)}\}$

Proof: $\mathrm{BTR}(m_S, m_B+1) \geq \mathrm{OPT}(m_S, m_B)$

- We will prove that
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	$\mathbb{E}[\mathrm{OPT}_{aug}] - \mathbb{E}[\mathrm{BTR}_{aug}]$	$\mathbb{E}[\mathrm{OPT}_{aug}] - \mathbb{E}[\mathrm{OPT}_{orig}]$
diff $\neq 0$ if …	$x^{(1)}\cdots\cdots x^{(m_S)} \overbrace{x^{(m_S+1)}\cdots\cdots}^{\text{a seller (seller } q)}$	$\overbrace{x^{(1)}\cdots\cdots x^{(m_S)}}^{\text{new buyer in top } m_S} x^{(m_S+1)}\cdots\cdots$
$\Pr[\text{diff} \neq 0]$	$m_S/(m_S+m_B+1)$ =	$m_S/(m_S+m_B+1)$
$\mathbb{E}[\mathrm{OPT}_{aug}]$	$\boxed{x^{(1)}\cdots\cdots x^{(m_S)}}\, x^{(m_S+1)}\cdots\cdots x^{(m_S+m_B)}$	
minus …	$\boxed{x^{(1)}\cdots\cdot}\,\overset{\text{buyer } q\ \downarrow}{x^{(\alpha)}}\,\boxed{\cdot\cdot\, x^{(m_S)}\ x^{(m_S+1)}}\cdots\cdots$	$\boxed{x^{(1)}\cdots}\,\overset{\text{new buyer}\ \downarrow}{x^{(\nu)}}\,\boxed{\cdots x^{(m_S)}\ x^{(m_S+1)}}\cdots\cdots$
diff$+x^{(m_S+1)}$	min of $q \geq 1$ vals $\sim U\binom{\{x^{(1)},\ldots,x^{(m_S)}\}}{q}$	val $\sim U\{x^{(1)},\ldots,x^{(m_S)}\}$

Proof: $\mathrm{BTR}(m_S, m_B+1) \geq \mathrm{OPT}(m_S, m_B)$

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	$\mathbb{E}[\mathrm{OPT}_{aug}] - \mathbb{E}[\mathrm{BTR}_{aug}]$		$\mathbb{E}[\mathrm{OPT}_{aug}] - \mathbb{E}[\mathrm{OPT}_{orig}]$
diff $\neq 0$ if ...	$x^{(1)}\ldots\ldots x^{(m_S)}\ \overbrace{x^{(m_S+1)}\ldots\ldots}^{\text{a seller (seller } q)}$		$\overbrace{x^{(1)}\ldots\ldots x^{(m_S)}}^{\text{new buyer in top } m_S}\ x^{(m_S+1)}\ldots\ldots$
$\Pr[\text{diff} \neq 0]$	$m_S/(m_S+m_B+1)$	$=$	$m_S/(m_S+m_B+1)$
$\mathbb{E}[\mathrm{OPT}_{aug}]$	$\boxed{x^{(1)}\ldots\ldots x^{(m_S)}}\ x^{(m_S+1)}\ldots\ldots x^{(m_S+m_B)}$		
minus ...	$\boxed{x^{(1)}\ldots.}\ \overset{\text{buyer } q}{\overset{\downarrow}{x^{(\alpha)}}}\ \boxed{..\ x^{(m_S)}\ x^{(m_S+1)}}\ldots\ldots$		$\boxed{x^{(1)}\ldots}\ \overset{\text{new buyer}}{\overset{\downarrow}{x^{(\nu)}}}\ \boxed{\ldots\ x^{(m_S)}\ x^{(m_S+1)}}\ldots\ldots$
diff$+x^{(m_S+1)}$	min of $q \geq 1$ vals $\sim U\left(\binom{\{x^{(1)},\ldots,x^{(m_S)}\}}{q}\right)$	$\leq$	val $\sim U\{x^{(1)},\ldots,x^{(m_S)}\}$

Result Summary and Open Questions

(#S,#B)	Condition	Sufficient #buyers* to add	Insufficient #buyers* to add
m_S, m_B	i.i.d. ($F_B = F_S$)	1	0 (MS'83)
m_S, m_B	any F_B, F_S	impossible, by $\Rightarrow$	any finite number
1,1	F_B FSD F_S	4	1
$1, m_B$	F_B FSD F_S	$4\sqrt{m_B}$	$\lfloor \log_2 m_B \rfloor$
m_S, m_B	F_B FSD F_S	$m_S(m_B + 4\sqrt{m_B})$	$\Uparrow$

* Exactly the same bounds also if adding sellers rather than buyers.

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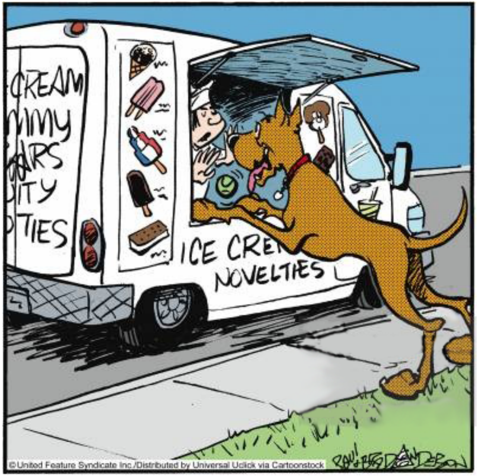
$\boxed{\textbf{Open: all gaps}}$



"Sorry, no trades. Cash only."

Questions?