SCHOOL CHOICE AND FAIRNESS PSET

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Problem 1. Give an example of an economy where the Boston Mechanism is not strategyproof.

Problem 2. The goal of this problem is to prove Theorem 1. This says that for a regular market (see Definition 4 on page 12), any set of budget probabilities can be implemented by DA with quotas and priorities. More formally, Given a market M and an arbitrary budget set probability matrix $y \in Y^M$, define the priority distribution G_t for each segment t as follows. For an agent i, sample a set $S_i \subseteq J$ according to the probability vector y_t (with probability y_{tS_i}). Define the agent's priority score for item j as

$$\pi_{ij} = \mathbb{1}_{(j \in S_i)} + \delta_i$$

, where $\delta_i \sim$ Uniform(0, 1). Define quota q_j as the left hand side of (11), which is the mass of agents assigned to j under budget set probabilities y. If M is regular, then the DA mechanism with priority distribution G and budget quota vector q implements y.

- a Let z be an n-dimensional vector of all 1's. Show that z is a fixed point of the DA operator (see $A.2$), ie. $DA(z) = z$.
- b Let X be the priority-based allocation mechanism associating each priority $\pi \in \Pi$ with the budget set
$$B_\pi^X = \{j \in J : j = 0 \text{ or } \pi_j \geq z_j\}$$
Show that for any agent segment t and any $S \subseteq J$ this set is equal to S with probability y_{tS}
- c Let $z' = z^{DA(M,G,q)}$ be the DA cutoff. This is the minimum element of the lattice of fixed points, $\{z' \in \Pi : DA(z') = z'\}$. Define X' to be DA mechanism with priority distribution G and quota q. This mechanism associates each priority $\pi \in \Pi$ with the budget set $B_\pi^{X'}$. Show that the budget set probabilities for the DA mechanism and our mechanism form part b are the same, ie. $y^X = y^{X'}$

Problem 3. We will show that in Example 1 of Section 5, the neighborhood assignment plan maximizes utilitarian welfare among all priority-based allocation mechanisms if and only if the following inequality holds:

$$E[u_0 - \alpha | u_0 \geq \alpha] \geq E[max(u_0, u_1) - \alpha | max(u_0, u_1) \geq \alpha],$$

where $u_0 \sim F_0$, $u_1 \sim F_1$ and $\alpha \sim H$ and the random variables are independent. On the other hand, the open enrollment plan (RSD) is optimal if and only if the above inequality is reversed.

- a Consider the LP setup of the priority allocation problem from lecture, and define the sets $S_0 = \{0\}, S_1 = \{0, 1\}, S_2 = \{0, 2\}, S_3 = \{0, 1, 2\}$. Show that without loss of generality it suffices to consider feasible solutions $y \in Y^M$ that are symmetric in the sense that $y_{1S_3} = y_{2S_3}, y_{1S_0} = y_{2S_0}, y_{1S_1} = y_{2S_2}, y_{1S_1} = y_{2S_2}$, and $y_{1S_2} = y_{2S_1}$.
- b Rewrite the LP using these symmetry conditions, with decision variable z_k replacing y_{1S_k} for $k = 1, 2, 3$. and $c = c_1 = c_2$ denoting the common capacity for each school.
- c Show that in any optimal solution z we must have $z_2 = 0$.
- d Plugging in $z_2 = 0$ yields that the feasible region is now a triangle. What are the three endpoints of that triangle?
- e Show that the vertex $(0, 0, 0)$ is never optimal, and translate the conditions where each of the non-zero vertices are optimal into the statement of the proposition.

Problem 4. Nguyen et al discussion: Do we agree with the issues pointed out by this paper about school choice? Is there anything we'd add or de-emphasize?

Problem 5. Hitzig discussion: What were your main takeaways of Hitzig's argument? Do you agree with it?