PROBLEMS FOR THE COURSE *INTRODUCTION TO MEAN CURVATURE FLOW*

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1. If there exists a coordinate neighborhood such that $\Gamma_{ij}^k = 0$ at each point, show that M is locally isometric to $\mathbb{R}^n$.
2. Let $(\mathbb{R}^n, g_0)$ be the (canonical) Euclidean metric $g_0(X,Y) = \Sigma x_i y_i$. Show that every isometry of this Riemannian manifold is a composition of a rotation, that is, an element in $O(n)$, and a translation.
3. Let $(S^n; g_0)$ be the metric on S^n induced from its usual embedding into $\mathbb{R}^{n+1}$: $x_1^2 + \ldots + x_{n+1}^2 = 1$.
   (a) What is the isometry group of g_0?
   (b) Show that if G is a group of isometries that acts freely on (S^{2n}, g_0), then $G = \{\pm \mathrm{Id}\}$.
4. Compute the Lie derivative of a vector field Y and of a 1-form ω, then compute the Lie derivative of a tensor T.
5. (a) Show that $\mathrm{div}(X)\mathrm{vol} := L_X\mathrm{vol} = d\iota_X\mathrm{vol}$, where $\iota_X T$ evaluates the tensor T on X in the first variable.
   (b) Prove the divergence theorem for Riemannian manifolds.
6. Let a metric be given by $dx^2 + f^2(x,y)dy^2$ with $f > 0$. Show that the curves $y = C$ are geodesics, in fact minimizing geodesics (for all values of x).
7. Describe the behavior of all geodesics on the following surfaces of revolution, where $f(s)$ is the distance to the axis of rotation:
   (a) $f(s) = e^s$
   (b) $f(s) = \cosh s$
   (c) $f(s) = \frac{1}{1+s^2}$.

   In each of the above examples describe all closed geodesics.
8. Let $A : M \to M$ be an isometry.
   (a) Show that $A(\exp_p v) = \exp_{A(p)}(dA(p)(v))$.
   (b) Show that $\mathrm{Fix}(A) = \{p \in M : A(p) = p\}$ is a union of totally geodesic submanifolds, that is, submanifolds $N \subset M$ with the property that if $p, q \in N$ are nearby points, then the unique minimizing geodesic $\bar{pq}$ of M is contained in N.
   (c) Give an example where $\mathrm{Fix}(A)$ has components of different dimensions.
9. Define a vector field X on a Riemannian manifold (M, g) to be a Killing field if its local flows preserve the metric. That is, if $U \subset X$ is any open subset so that the flow $U_t := \phi_t^X(U)$ is well defined for $|t| < \varepsilon$ then $(phi_t^X) * (g|_{U_t}) = g|_U$.

   Show that a vector field X on a Riemannian manifold (M, g) is a Killing field if and only if $L_X g = 0$.

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10. Show that on a compact Riemannian manifold, the existence of a non-trivial Killing field is equivalent to the existence of a (non-trivial) isometric S^1-action.
11. (a) (Berger) Show that on a closed, even-dimensional manifold of positive sectional curvature, any isometric S^1-action admits a fixed point.
    (b) (Sugahara) Show that on a closed, odd-dimensional manifold of positive sectional curvature, any isometric T^2-action has a circle orbit or admits a fixed point.
12. Show that for a compact hypersurface in $M^n \subset \mathbb{R}^{n+1}$, there exists a point p where $\sec(M) > 0$ for all 2-planes in T_pM. What can you say if the codimension is bigger than one?
13. Show that on a 3-dimensional manifold the Ricci curvature determines the sectional curvature. Show that an Einstein metric on a 3-dimensional manifold has constant sectional curvature.
14. Calculate the mean curvature of the round sphere, $S^n(1) \subset \mathbb{R}^{n+1}$.
15. (Exercise 1.1.2 from Mantegazza) Show that if the hypersurface $M \subset \mathbb{R}^{n+1}$ is locally the graph of a function $f : \mathbb{R}^n \to \mathbb{R}$, that is, $\phi(x) = (x, f(x))$, then

$$g_{ij} = \delta_{ij} + f_i f_j, \quad \nu = -\frac{(\nabla f, -1)}{\sqrt{1+|\nabla f|^2}}, \quad h_{ij} = \frac{\mathrm{Hess}_{ij} f}{\sqrt{1+|\nabla f|^2}}$$

$$H = \frac{\Delta f}{\sqrt{1+|\nabla f|^2}} - \frac{\mathrm{Hess} f(\nabla f, \nabla f)}{\sqrt{1+|\nabla f|^2}^3} = \mathrm{div}\left(\frac{\nabla f}{\sqrt{1+|\nabla f|^2}}\right),$$

where $f_i = \partial_i f$, ν is the normal vector to M, h_{ij} is the second fundamental form of M (whose trace is the mean curvature H).
16. (Exercise 1.1.3 from Mantegazza) Show that if the hypersurface $M \subset \mathbb{R}^{n+1}$ is locally the zero set of a smooth function $f : \mathbb{R}^n \to \mathbb{R}$, with $\nabla f \neq 0$ on such a level set, then we have

$$H = \frac{\Delta f}{|\nabla f|} - \frac{\mathrm{Hess} f(\nabla f, \nabla f)}{|\nabla f|^3} = \mathrm{div}\left(\frac{\nabla f}{|\nabla f|}\right).$$

17. (Exercise 1.8 from Haslhofer) Let $M_t^n \subset \mathbf{R}^{n+1}$ be a mean curvature flow of surfaces, and let $\lambda > 0$. Let $M_{t'}^\lambda$ be the family of surfaces obtained by the parabolic rescaling $x' = \lambda x$, $t' = \lambda^2 t$, that is, let $M_{t'}^\lambda = \lambda M_{\lambda^{-2}t'}$. Show that $M_{t'}^\lambda$ again solves the mean curvature flow.
18. Consider the family of curves in $\mathbb{R}^2$ given by $M_t = \mathrm{graph}(u(p,t))$, where $u(p,t) = \log\cos p + t$ for $p \in (-\pi/2, \pi/2)$. This family of curves is known as the *grim reaper*. How does this family of curves evolve under the curve shortening flow?
19. Let (M_t) be a family of concentric n-spheres in R^{n+1}, that is,

$$M_t = \partial B^{n+1}(r(t)).$$

Show that a solution to the mean curvature flow for this family of embedded hypersurfaces exists for $t \in (-\infty, \sqrt{r(0) - 2n})$. How does this family of hypersurfaces evolve under the mean curvature flow?
20. Let (M_t) be a family of spherical cylinders, that is,

$$M_t = \partial B^{n+1-k}(r(t)) \times \mathbb{R}^k$$

for $0 \leqslant k \leqslant n$. Show a solution to the mean curvature flow for this family of embedded hypersurfaces exists for $t \in (-\infty, r(0)/2(n-k))$. How does this family of hypersurfaces evolve under the mean curvature flow?