PROBLEM SET FOR THE COURSE *SINGULARITY ANALYSIS FOR THE MEAN CURVATURE FLOW*

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1. Prove the following evolution equations on a mean curvature flow of hypersurfaces $\Sigma_t \subset \mathbb{R}^{n+1}$:
   (i) $\left(\frac{d}{dt} - \Delta\right) |x|^2 = -2n$,
   (ii) $\left(\frac{d}{dt} - \Delta\right) H = H|A|^2$,
   (iii) $\left(\frac{d}{dt} - \Delta\right) |A|^2 = -2|\nabla A|^2 + 2|A|^4$,
   (iv) $\left(\frac{d}{dt} - \Delta\right) |\nabla^m A|^2 \leq -2|\nabla^{m+1} A|^2 + C(m,n) \sum_{i+j+k=m} |\nabla^i A||\nabla^j A||\nabla^k A||\nabla^m A|$.

2. Let $\{\Sigma_t\}_{t\in[0,T)} \subset \mathbb{R}^{n+1}$ be a mean curvature flow starting from a closed hypersurface Σ_0. Show that
   (i) If $H \geq 0$ on Σ_0, then $H \geq 0$ on Σ_t for $t \in [0,T)$.
   (ii) If $\mathbf{x} \cdot \nu \leq 0$ on Σ_0, then $2tH - \mathbf{x} \cdot \nu \geq 0$ on Σ_t for $t \in [0,T)$. Here ν is the unit normal of Σ_t.

3. If $\{\Sigma_t\}$ is an n-dimensional mean curvature flow in $B^{n+1}_\rho(x_0) \times (t_0 - \rho^2, t_0)$ such that
$$|A(x)|^2 \leq \frac{C_0}{\rho^2}$$
for $x \in \Sigma_t \cap B^{n+1}_\rho(x_0)$ and $t \in (t_0 - \rho^2, t_0)$, then, for each $m \geq 1$, there is a constant $C_m = C_m(n, C_0)$ so that
$$|\nabla^m A(x)|^2 \leq \frac{C_m}{\rho^{2(m+1)}}$$
for $x \in \Sigma_t \cap B^{n+1}_{\rho/2}(x_0)$ and $t \in (t_0 - \rho^2/4, t_0)$.

4. Suppose $\{\Sigma_t\}_{t\in[0,T)}$ is a mean curvature flow on a maximal time interval $[0,T)$ starting from a closed hypersurface Σ_0. Prove that
$$\max_{\Sigma_t} |A|^2 \geq \frac{C}{T-t}$$
for some constant $C > 0$.

5. Let $\mathcal{M} = \{\Sigma_t\}_{t\in[0,T)}$ be a mean curvature flow of hypersurfaces in $\mathbb{R}^{n+1}$. If $X_0 = (x_0, t_0)$ is a spacetime point of $\mathcal{M}$ such that
$$\Theta(\mathcal{M}, X_0, r) = 1$$
for all $r > 0$, then $\mathcal{M}$ is a static hyperplane through X_0. Here
$$\Theta(\mathcal{M}, X_0, r) = (4\pi r^2)^{-\frac{n}{2}} \int_{\Sigma_{t_0 - r^2}} e^{-\frac{|x-x_0|^2}{4r^2}}.$$

6. Let $\{\Sigma_t\}_{t<0}$ be a mean curvature flow of hypersurfaces in $\mathbb{R}^{n+1}$. Show that $\Sigma_t = \sqrt{-t}\,\Sigma_{-1}$ for all $t<0$ if and only if $\vec{H} - \frac{x^\perp}{2t} = 0$ on Σ_t for all $t<0$.

7. Let $\{\Sigma_t\}$ be an n-dimensional mean curvature flow in $B^{n+1}_{\sqrt{4n}\rho}(x_0)\times(t_0-\rho^2,t_0)$.
   (i) If $\eta^\rho_{X_0}(x,t) = \left(1 - \frac{|x-x_0|^2+2n(t-t_0)}{\rho^2}\right)^3_+$, then
$$\left(\frac{d}{dt} - \Delta\right)\eta^\rho_{X_0} \le 0.$$
   (ii) Show that
$$\frac{d}{dt}\int_{\Sigma_t}\eta^\rho_{X_0}\Phi_{X_0} \le -\int_{\Sigma_t}\left|\vec{H} + \frac{(x-x_0)^\perp}{2(t_0-t)}\right|^2\eta^\rho_{X_0}\Phi_{X_0}$$
   where $\Phi_{X_0}(x,t) = (4\pi(t_0-t))^{-n/2}\exp(\frac{|x-x_0|^2}{4(t-t_0)})$.

8. For a hypersurface $\Sigma\subset\mathbb{R}^{n+1}$, the Colding-Minicozzi entropy of Σ is given by
$$\lambda(\Sigma) = \sup_{x_0\in\mathbb{R}^{n+1},t_0>0}(4\pi t_0)^{-\frac{n}{2}}\int_\Sigma e^{-\frac{|x-x_0|^2}{4t_0}}.$$
Show that
   (i) $\lambda(\rho\Sigma + y) = \lambda(\Sigma)$ for any $\rho>0$ and $y\in\mathbb{R}^{n+1}$.
   (ii) If $\{\Sigma_t\}_{t\in[0,T)}\subset\mathbb{R}^{n+1}$ is a mean curvature flow starting from a closed hypersurface, then $t\mapsto\lambda(\Sigma_t)$ is nonincreasing.

9. Verify certain hyperplanes, round sphere, and generalized cylinders are self-shrinkers. Specifically,
   (i) Find all hyperplanes that are self-shrinkers.
   (ii) If $\mathbb{S}^n_r(x_0)$ is the round n-sphere with radius r and center x_0, find all (r,x_0) so that $\mathbb{S}^n_r(x_0)$ is a self-shrinker.
   (iii) For $0<k<n$, if $\mathbb{S}^k_r(x_0)\times\mathbb{R}^{n-k}$ is a generalized cylinder, find all (r,x_0) so that $\mathbb{S}^k_r(x_0)\times\mathbb{R}^{n-k}$ is a self-shrinker.

10. If $\gamma\subset\mathbb{R}^2$ is a simple self-shrinking curve, prove that
$$x\cdot\nu e^{-\frac{|x|^2}{4}}$$
is constant on γ, where ν is the unit normal on γ.

11. Prove that, on a hypersurface $\Sigma\subset\mathbb{R}^{n+1}$,
$$\left(1+\frac{2}{n+1}\right)|\nabla|A||^2 \le |\nabla A|^2 + \frac{2n}{n+1}|\nabla H|^2.$$

12. Let $\Sigma\subset\mathbb{R}^{n+1}$ be a strictly mean convex (i.e., $H>0$) self-shrinker with at most polynomial area growth. If $f\colon\Sigma\to\mathbb{R}$ satisfies
$$\int_\Sigma(f^2+|\nabla f|^2)e^{-\frac{|x|^2}{4}} < \infty$$
then
$$\int_\Sigma f^2(|A|^2+|\nabla\log H|^2)e^{-\frac{|x|^2}{4}} \le \int_\Sigma(4|\nabla f|^2+2f^2)e^{-\frac{|x|^2}{4}}.$$

13. Let $\Sigma \subset \mathbb{R}^{n+1}$ be a self-shrinker with at most polynomial area growth. Suppose Σ is given by the graph of a smooth function $u\colon \mathbb{R}^n \to \mathbb{R}$, i.e.,

$$\Sigma = \mathrm{Graph}_u = \{(y, u(y)) \mid y \in \mathbb{R}^n\}.$$

In the following, we will show Σ must be flat.

(i) If $\nu = (\nu_1, \ldots, \nu_{n+1})$ is the unit normal on Σ, then $|\nu_{n+1}| > 0$ and

$$\Delta \nu_{n+1} - \frac{x}{2} \cdot \nabla \nu_{n+1} + |A|^2 \nu_{n+1} = 0.$$

(ii) If $\phi\colon \Sigma \to \mathbb{R}$ is smooth with compact support, then

$$\int_\Sigma |A|^2 \phi^2 e^{-\frac{|x|^2}{4}} \leq \int_\Sigma |\nabla \phi|^2 e^{-\frac{|x|^2}{4}}.$$

(Thus, choosing a sequence of cutoffs $\phi = \phi_i$ converging to 1, the monotone convergence theorem implies

$$\int_\Sigma |A|^2 e^{-\frac{|x|^2}{4}} \leq 0.$$

Hence $|A| = 0$, that is, Σ is flat.)

14. For $n \geq 2$, let $\Sigma \subset \mathbb{R}^{n+1}$ be a strictly mean convex (i.e., $H > 0$) self-shrinker with at most polynomial area growth. We proved in lecture that $|A| = \beta H$ for some constant $\beta > 0$. In the following, we will show this implies that Σ is a generalized cylinder.

(i) Show that $|\nabla |A||^2 = |\nabla A|^2$.

(ii) If the rank of A is at least two at some point $p \in \Sigma$, show that $\nabla A(p) = 0$ and, indeed, this holds at each point of Σ. (Thus, by a result of Lawson, it follows that Σ is a generalized cylinder.)

(iii) Suppose the rank of A is equal to 1 at each point of Σ. Show that there is a $(n-1)$-dimensional vector space V so that Σ is invariant under the translations in V.

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