Introduction to minimal surfaces

Lecture 1 Problems

1. Let Σ be an immersed submanifold of dimension k in $\mathbb{R}^n$.

   (a) Show that Σ is minimal if and only if the coordinate functions of Σ in $\mathbb{R}^n$ are harmonic, $\Delta_\Sigma x^i = 0$, $i = 1, \ldots, n$:

   i. Using the first variation formula. (*Hint:* Consider the variation $X = \eta E_j$ where $\eta \in C_c^\infty(\Sigma)$, where $E_1, \ldots, E_n$ are the standard basis vectors of $\mathbb{R}^n$.)

   ii. By showing that $\Delta_\Sigma x = H$, where x is the position vector and H is the mean curvature vector of Σ in $\mathbb{R}^n$.

   (*Recall:* If (M, g) is Riemannian manifold, the Laplace operator is the trace of the Hessian,

   $$\Delta_M f = \operatorname{Tr}_g \operatorname{Hess} f = e_i\, e_i\, f - (\nabla_{e_i} e_i)\, f$$

   where $e_1, \ldots, e_m$ is a local orthonormal frame on M.)

   Conclude that there are no closed (compact without boundary) minimal submanifolds in $\mathbb{R}^n$.

   (b) If Σ is minimal, show that $\Delta |x|^2 = 2k$ and

   $$k\,|\Sigma| = \int_{\partial\Sigma} x \cdot \nu \, dV_{\partial\Sigma},$$

   where $|\Sigma|$ denotes the volume of Σ and $\nu_{\partial M}$ is the outward unit conormal of Σ along $\partial\Sigma$.

2. (Convex Hull Property) Let Σ^k be a compact minimal submanifold in $\mathbb{R}^n$ with smooth boundary. Show that Σ is contained in the convex hull of its boundary,

   $$\Sigma \subset \operatorname{Conv}(\partial\Sigma) = \cap\{H : H \text{ is a half space of } \mathbb{R}^n \text{ containing } \partial\Sigma\}$$

3. Let $u : (M, g) \to (N, h)$ be a C^1 map between Riemannian manifolds. The energy functional is defined by

   $$E(u, g) = \int_M \|du\|_g^2 \, dV_g$$

   where $\|du\|_g^2 = \operatorname{Tr}_g(u^* h) = g^{\alpha\beta}(x) h_{ij}(u(x)) \frac{\partial u^i}{\partial x^\alpha} \frac{\partial u^j}{\partial x^\beta}$ and dV_g is the Riemannian volume form of (M, g). Show that if M is a surface ($\dim M = 2$) then the energy is conformally invariant:

   (a) If $\tilde{g} = \lambda g$ for some function $\lambda : M \to (0, \infty)$, then

   $$E(u, \tilde{g}) = E(u, g).$$

   (b) If $F : (\tilde{M}, \tilde{g}) \to (M, g)$ is conformal (i.e. $F^* g = \lambda \tilde{g}$), then

   $$E(u \circ F, \tilde{g}) = E(u, g)$$

4. Find all connected minimal surfaces of revolution in $\mathbb{R}^3$.

   [*Hint:* Consider the surface of revolution generated by the curve $\alpha(t) = (r(t), t)$. Write the ordinary differential equation that expresses the condition that the mean curvature of the surface equals zero.]

5. Let $\mathbb{S}^n$ be the unit sphere in $\mathbb{R}^{n+1}$ with the induced metric. Show that a k-dimensional submanifold Σ of the sphere $\mathbb{S}^n$ is minimal if and only if the coordinate functions $x^1, \ldots, x^{n+1}$ are eigenfunctions of the Laplacian on Σ, with eigenvalue k,

$$\Delta_\Sigma x^i + k x^i = 0, \qquad i = 1, \ldots, n+1.$$

   [*Hint:* Work in an adapted local orthonormal frame $e_1, \ldots, e_{n+1}$, such that $e_1, \ldots, e_k$ are tangent to Σ, $e_{k+1} = x$ is the outward unit normal to $\mathbb{S}^n$, and $e_{k+2}, \ldots, e_{n+1}$ are normal to Σ and tangent to $\mathbb{S}^n$.]