Introduction to minimal surfaces

Problem Set 2

1. (Courant-Lebesgue Lemma) Suppose $u : D \to \mathbb{R}^n$, $u \in W^{1,2}(D)$, $E(u) \leq \frac{K}{2}$ for some constant K. Using the steps below, show that for any $\delta \in (0,1)$, there exists $\rho \in [\delta, \sqrt{\delta}]$ with
$$\ell(u(C_\rho))^2 \leq \frac{4\pi K}{|\log \delta|}$$
where ℓ denotes the length, and $C_\rho = \{x \in D : |x-p| \leq \rho\}$ is the circle of radius ρ about a fixed point $p \in D$.

   (a) Let $P(r) = r \int_{C_r} \|\nabla u\|^2 \, ds$. Use the mean value theorem for integrals for the measure $d(\log r)$ to show that there exists $\rho \in [\delta, \sqrt{\delta}]$ such that
$$P(\rho) \leq \frac{2K}{|\log \delta|}.$$

   (b) Use Hölder's inequality to show that $P(\rho)$ dominates the length of the image curve:
$$\ell(u(C_\rho))^2 \leq 2\pi P(\rho).$$

   (c) Combine (a) and (b) to conclude the result.

2. Let Σ be an oriented surface and let $u : (\Sigma, g) \to (N, h)$ be a smooth map. Define the *Hopf differential*
$$\phi(z) = h(u_z, u_z)\, dz^2 = (\|u_x\|^2 - \|u_y\|^2 - 2i\langle u_x, u_y\rangle)\, dz^2$$
where $z = x + iy$ are local complex coordinates on Σ. Show that if u is harmonic then the Hopf differential is holomorphic.

   Recall that a map is harmonic if it is a critical point of the energy functional E. In local coordinates, the harmonic map equation is
$$\Delta_\Sigma u^i + g^{\alpha\beta}\Gamma^i_{jl}(u(x))\, \frac{\partial u^j}{\partial x^\alpha} \frac{\partial u^l}{\partial x^\beta} = 0 \quad i = 1, \ldots, n$$
where Γ^i_{jl} are the Christoffel symbols for the Levi-Civita connection on N.
(You can derive this Euler-Lagrange equation if you have time.)

   Note that in normal coordinates centered at $u(x)$ the harmonic map equation at x reduces to $\Delta_\Sigma u^i = 0$, $i = 1, \ldots, n$, since $\Gamma^i_{jl}(u(x)) = 0$.

   Remark: From the definition, we see that u is (weakly) conformal if and only if the Hopf differential vanishes everywhere.

3. Show that any harmonic map from S^2 to an arbitrary Riemannian manifold must be conformal, and $u(S^2) = \Sigma$ is a minimal surface.

4. (Monotonicity Theorem) Let Σ be a k-dimensional minimal submanifold of $\mathbb{R}^n$ and let $p \in \mathbb{R}^n$. Show that

$$\Theta(M, p, r) := \frac{|\Sigma \cap B(p, r)|}{\omega_k r^k}$$

is an increasing function of r for $0 < r \leq \text{dist}(p, \partial\Sigma)$, using the steps outlined below. Here ω_k is the volume of the unit ball in $\mathbb{R}^k$.

[Thus $\Theta(M, p, r)$, called the *density ratio* of Σ in $B(p, r)$, is the volume of $\Sigma \cap B(p, r)$ divided by the area of the cross-sectional k-disk in $B(p, r)$.)]

We may assume that $p = 0$ and let $\Sigma_r = \Sigma \cap B(0, r)$, so $\partial\Sigma_r = \Sigma \cap \partial B(0, r)$. Let $A(r) = |\Sigma_r|$ and $L(r) = |\partial\Sigma_r|$. Then

$$A'(r) \geq L(r). \tag{1}$$

This follows from the "coarea formula" applied to the function $x \in \Sigma \to |x|$. Intuitively, (in the case $k = 2$ for simplicity) $A(r + dr) \setminus A(r)$ is a thin ribbon of surface: the length of the ribbon is $L(r)$ and the width is $\geq dr$. (The width is equal to dr at a point $p \in \partial\Sigma_r$ if and only if Σ is orthogonal to $\partial B(0, r)$ at p.) Hence $A(r + dr) - A(r) \geq L(r)dr$.

To prove the monotonicity theorem, use 1(b) from yesterday's problems together with equation (1) to show that

$$(r^{-k}A)' \geq 0.$$