Introduction to minimal surfaces

Problem Set 3

1. (Second variation formula for minimal surfaces for normal variations)

   Let Σ be a k-dimensional minimal submanifold of a Riemannian manifold (M, g).

   Let $F : \Sigma \times (-\epsilon, \epsilon) \to M$ be a variation of Σ with compact support and fixed boundary. That is, $F =$Id outside a compact set,

$$F(x, 0) = x,$$

   and for all $x \in \partial\Sigma$,

$$F(x, t) = x$$

   Set $\Sigma_t := F(\Sigma, t)$. Assume that F is a normal variation; that is the variation field $V = dF(\frac{\partial}{\partial t})|_{t=0}$ is normal to Σ.

   Following the steps outlined below, show that

$$\begin{aligned}\left.\frac{d^2}{dt^2}\right|_{t=0} \mathrm{Vol}(\Sigma_t) &= \int_\Sigma \left(\|\nabla^\perp V\|^2 - \|A^V\|^2 - \langle R(V), V\rangle \right) dV_g \\ &= \int_\Sigma \left(\sum_{i=1}^k \|\nabla^\perp_{e_i} V\|^2 - \sum_{i,j=1}^k \langle \nabla_{e_i} e_j, V\rangle^2 - \sum_{i=1}^k \langle R(V, e_i)e_i, V\rangle \right) dV_g\end{aligned}$$

   where $\nabla^\perp$ denotes the normal connection, and $e_1, \ldots, e_k$ is a local orthonormal frame tangent to Σ.

   Letting $g_{ij}(t) = \langle F_{x^i}, F_{x^j}\rangle$, recall from our derivation of the first variation formula that $\mathrm{Vol}(\Sigma_t) = \int_\Sigma \nu(x, t)\, dV_g$, where $\nu(x, t) = \sqrt{\det g(t)}/\sqrt{\det g(0)}$, and

$$\frac{d}{dt}\nu(x, t) = \frac{1}{2}\nu(x, t)\,\mathrm{Tr}_g(g') = \frac{1}{2}\nu(x, t) \sum_{i,j=1}^k g^{ij}(t) g'_{ij}(t)$$

   To evaluate at a fixed point $x \in \Sigma$, choose coordinates such that $g_{ij}(0)(x) = \delta_{ij}$.

   (a) Show that

$$\left.\frac{\partial^2 \nu}{\partial t^2}\right|_{t=0} = \frac{1}{4}\left(\mathrm{Tr}_g(g')\right)^2 + \frac{1}{2}\mathrm{Tr}_g(g'') - \frac{1}{2}\sum_{i,j=1}^k (g'_{ij})^2$$

   (b) Show that $\sum_{i,j}^k (g'_{ij})^2 = 4\|A^V\|^2$.

   (c) Show that

$$\mathrm{Tr}_g(g'') = 2\langle R(V), V\rangle + 2\,\mathrm{div}_\Sigma(\nabla_{F_t} F_t) + 2\|A^V\|^2 + \|\nabla^\perp V\|^2$$

   (d) Combine (a), (b), (c) to obtain the second variation formula above.

2. Show that an equatorial $S^n \subset S^{n+1}$ is a minimal hypersurface of index 1 and nullity $n+1$.

   Hint: Use that the first eigenvalue of the Laplace operator of the sphere $\lambda_1(\Delta_{S^n}) = 0$ with eigenspace the constant functions and the second eigenvalue $\lambda_2(\Delta_{S^n}) = 2$ with eigenspace spanned by the coordinate functions $x^1, \ldots, x^{n+1}$ restricted to the sphere. Assume this for this problem. For a reference, see Chapter IV of *Riemannian Geometry* by Gallot, Hulin, Lafontaine.

3. Let Σ^n be a (two-sided) minimal hypersurface in S^{n+1}.

   (a) Let ν be a unit normal to Σ in $S^{n+1} \subset \mathbb{R}^{n+2}$. Show that the components $\nu^1, \ldots, \nu^{n+2}$ of ν are eigenfunctions of the Jacobi operator L with eigenvalue $-n$,

   $$L\nu^i - n\nu^i = 0$$

   (b) Show that if Σ is not an equator then $\text{index}(\Sigma) \geq n+3$.

   Hint: Use that the lowest eigenvalue of the Jacobi operator must be simple (i.e. the eigenspace has dimension 1). This is a standard fact about self-adjoint elliptic operators (see for example Gilbarg & Trudinger).

Remarks: It is known that the Clifford torus $S^1(\frac{1}{\sqrt{2}}) \times S^1(\frac{1}{\sqrt{2}})$ is the unique minimal surface in S^3 of index 5. This was used in the proof of the Willmore conjecture.

The question of what the minimal hypersurfaces $\Sigma^n \subset S^{n+1}$, $n \geq 3$, with $\text{index}(\Sigma) = n+3$ are is open. It is known that the Clifford hypersurfaces

$$S^p\left(\sqrt{\frac{p}{p+q}}\right) \times S^q\left(\sqrt{\frac{q}{p+q}}\right) \subset S^{p+q+1}$$

have index $n+3$, and it is conjectured that they are the only ones of index $n+3$.