PROBLEM SETS

The problems marked with (†) will be used elsewhere, either in the lectures or for other problems in the problem sets. The problems marked with ($\star$) are suitable for presentations, and some of them can be effectively broken down into smaller parts for group presentations.

Caution: Some problems are really challenging! While the problem sets are organized according to the order of lectures, you are not required to work on them in the prescribed order; feel free to explore other problem sets and focus on the ones that intrigue you the most. Furthermore, the problems are organized based on their similarity, resulting in multiple subproblems for each main problem. It is worth noting that these subproblems can be quite long.

0.1. **Problem Set 1.**

Problem 1. (†) Suppose g is a rotationally symmetric metric on $[a,\infty)\times S^{n-1}$ in the sense that

$$g = \frac{1}{f(r)}dr^2 + r^2 d\Omega^2.$$

where $d\Omega^2$ is the standard unit sphere metric on S^{n-1}.

(1) Compute the second fundamental form and mean curvature of the r-level set.
(2) Show the scalar curvature of g is given by

$$R_g = \frac{n-1}{r^2}\left((n-2)(1-f(r)) - rf'(r)\right).$$

(Hint: One way is to use the second variational formula of the volume of the r-level set.)

Problem 2. (†, $\star$) Let M be a closed manifold of dimension $n \geq 3$. Then the following holds:

(1) M admits a metric of positive (resp. zero, negative) scalar curvature conformal to g if and only if $\lambda_1(g) > 0$ (resp. $\lambda_1(g) = 0$, $\lambda_1(g) < 0$).
(2) If g is a Riemannian metric on M such that $R_g \geq 0$ everywhere and $R_g > 0$ somewhere, then $\lambda_1(g) > 0$.

Problem 3. Let (M, g) be a 3-dimensional manifold.

(1) If $\mathrm{Ric}_g > 0$, then (M, g) does not contain any closed stable minimal surface.
(2) If $\mathrm{Ric}_g \geq 0$ and Σ is a closed stable minimal torus, then Σ is totally geodesic, $\mathrm{Ric}_g(\nu,\nu) = 0$ along Σ, and Σ is a flat torus.

In the next problem, we will prove the following theorem originally due to Cai and Galloway: If (M^3, g) is a 3-manifold with $R_g \geq 0$ and Σ is a torus that is locally area-minimizing in M, then M is flat in a neighborhood of Σ.

Problem 4. (†, $\star$) Let (M, g) be a 3-dimensional manifold with $R_g \geq 0$. Let Σ be a stable minimal torus.

(1) Show that Σ is totally geodesic, $\mathrm{Ric}_g(\nu,\nu)=0$, and the induced metric on Σ is flat.

(2) Consider the map $\Psi : \mathcal{C}^{2,\alpha}(\Sigma)\times\mathbb{R}\to\mathcal{C}^{0,\alpha}(\Sigma)\times\mathbb{R}$ defined by

$$\Psi(u,a)=\left(H_{\Sigma(u)}-a,\int_\Sigma u\right)$$

where $\Sigma(u)=\exp(u\nu)$ denotes the normal graph on Σ. Show that Ψ is local diffeomorphism at $(u,a)=(0,0)$. (Hint: show the linearization of Ψ at $(0,0)$ is an isomorphism and apply the Inverse Function Theorem.)

(3) Let $(u(t),a(t))$ solve $\Psi(u(t),a(t))=(0,t)$. Show that $a'(0)=0$, and $u'(0)>0$. In particular, $u(t)$ is strictly increasing in t for all $|t|$ small, and thus $\Sigma(u(t))$ forms a foliation of constant mean curvature tori near Σ.

(4) Assume Σ is locally area minimizing, i.e. Σ has area less or equal to that of all nearby surfaces. Show that $a(t)\equiv 0$ for $|t|$ small and each $\{\Sigma_t\}$ is locally area minimiziing. Then show (M,g) splits as $(-\epsilon,\epsilon)\times\Sigma$ with the metric dt^2+g_Σ where g_Σ is the induced metric on Σ.

0.2. **Problem Set 2.**

Problem 5. ($\star$) Suppose (M^3,g) has $R_g\geq 2$. Let Σ be a closed stable minimal surface.

(1) Each component of Σ has area $\leq 4\pi$.

(2) If Σ has area 4π, then Σ is totally geodesic, $R=2$ and $\mathrm{Ric}(\nu,\nu)=0$ along Σ, and Σ with the induced metric is isometric to a round sphere of radius 1.

Problem 6 (Birkhoff's theorem)**.** Using Problem 1, show that the only rotationally symmetric metric with zero scalar curvature defined on $[a,\infty)\times S^{n-1}$ is given by

$$g=\frac{1}{f(r)}dr^2+r^2d\Omega^2$$

where $f(r)=1-\frac{2m}{r^{n-2}}$. By changing of coordinates, the metric can be rewritten as another coordinate presentation of the Schwarzschild metric $g_m=\left(1+\frac{m}{2|x|^{n-2}}\right)^{\frac{4}{n-2}}g_{\mathbb{E}}$.

Problem 7. ($\star$) Consider the Schwarzschild metric $g_m=\left(1+\frac{m}{2|x|}\right)^4 g_{\mathbb{E}}$.

(1) Find the area $A(r)$ of $S_r=\{x:|x|=r\}$ in the metric g_m. For $m>0$, show that $A(r)$ has a global minimum at $\Sigma_0:=\{r=\frac{m}{2}\}$.

(2) For $m>0$, show that $r\to\frac{m^2}{4r}$ induces an isometry of g_m fixing Σ_0.

(3) When $m<0$, show that $A(r)\to 0$ as $r\to-\frac{m}{2}$. Also show that a radial geodesic from $r=r_0>-\frac{m}{2}$ to $r=-\frac{m}{2}$ has finite length. Show that the Schwarzschild metric with $m<0$ cannot be completed by adding in a point.

Problem 8. ($\dagger$) Let (M,g) be a 3-dimensional asymptotically flat manifold. If $\bar g=u^4g$ for some $u>0$ satisfying $u(x)=1+O_1(|x|^{-1})$, then

$$m(\bar g)=m(g)-\frac{1}{2\pi}\lim_{r\to\infty}\int_{S_r}\nu(u)\,d\sigma \tag{1}$$

where $\nu = \frac{x}{r}$ is the outward unit normal to S_r with respect to $g_{\mathbb{E}}$. Use the formula (1) to verify that the Schwarzschild metric g_m has ADM mass m.

0.3. **Problem Set 3.**

Problem 9. $(\dagger, \star)$ Let U be an open subset of Euclidean $\mathbb{R}^n$. For u locally integrable in U, denote by u^ϵ the standard mollification.

(1) If u is harmonic in U, then $u^\epsilon(x) = u(x)$ for all $x \in U_\epsilon$.
(2) If u is superharmonic in U, then u^ϵ is superhamonic in U_ϵ.

Problem 10. $(\dagger, \star)$ Let (Σ, g) be a complete surface that has quadratic area growth; namely there is a constant $C > 0$ such that for any geodesic ball B_s^Σ,

$$\mathrm{Area}(B_s^\Sigma) \leq Cs^2.$$

Construct a sequence of functions f_k that converge to 1 at each point and the Dirichlet energy $\int_\Sigma |\nabla f_k|^2 \to 0$ as $k \to \infty$. (They are called the *logarithmic cut-off functions.*)

For the next problem, we give the following definition.

Definition 0.1. A manifold Σ is said to be *parabolic* if any positive superharmonic function is constant.

Problem 11. $(\star)$ Let (M, g) be a complete manifold with quadratic area growth. Then M is parabolic. (Hint: If $u > 0$ is superharmonic, consider $w = \log u$. Multiply the equation for Δw by logarithmic cut-off functions and integrate.)

0.4. **Problem Set 4.**

In this problem set, we break down the proof of the rigidity of the positive mass theorem into the following problems.

Theorem 0.2. *Let (M, g) be asymptotically flat with $R_g \geq 0$. If $m(g) = 0$, then (M, g) is isometric to $(\mathbb{R}^n, g_{\mathbb{E}})$.*

In the first problem, we will show that $R_g \equiv 0$.

Problem 12. $(\star)$ Let (M, g) be asymptotically flat with $m(g) = 0$. Suppose $R_g \geq 0$ everywhere and $R_g > 0$ somewhere.

(1) Show that there is a scalar function $u > 0$ with $u \to 1$ as $|x| \to \infty$ solving $8\Delta_g u - R_g u = 0$.
(2) Define $\hat{g} := u^4 g$. Show that $R_g = 0$ and use Problem 8 to show that $m(\hat{g}) < 0$.

In the next problem, we will show that $\mathrm{Ric}_g \equiv 0$.

Problem 13. $(\star)$ Let (M, g) be asymptotically flat with $m(g) = 0$. Suppose $R_g = 0$. Let h be a compactly supported, symmetric $(0, 2)$-tensor. Consider the family of metrics $g(t) = g + th$.

(1) Show that for $|t|$ small, there exists a unique solution $u_t > 0$ with $u_0 \equiv 1$ and $u_t \to 1$ as $|x| \to \infty$ such that
$$-8\Delta_{g(t)} u_t + R_{g(t)} u_t = 0.$$
Consequently, the metric $\hat{g}(t) = u_t^4 g(t)$ has zero scalar curvature with $\hat{g}(0) = g$.

(2) Using Problem 8, show that the ADM mass of $\hat{g}(t)$ satisfies
$$m(\hat{g}(t)) = m(g) - \frac{1}{16\pi} \int_M R_{g(t)} u_t \, dv$$
and thus
$$\left.\frac{d}{dt}\right|_{t=0} m(\hat{g}(t)) = \frac{1}{16\pi} \int_M \mathrm{Ric}_g \cdot h \, dv.$$

(3) Using positivity of the ADM mass, show that g has zero Ricci curvature.

Problem 14. ($\star$) Let (M, g) be a 3-dimensional asymptotically flat manifold with $\mathrm{Ric}_g \equiv 0$. Show that M must diffeomorphic to $\mathbb{R}^3$ and (M, g) must be isometric to the Euclidean space.