Exercise Sheet 1
Convexity, Martingales and Concentration inequalities

We do not expect you to solve all the exercises in this sheet. You can use these exercises as useful references and perhaps to practice in the future.

Easy

Exercise 1

Goal of this exercise is to become familiar with the concept of convexity and subdifferential.

1. Let $A, B \subset \mathbb{R}^n$. Show that the Minkwoski sum $A + B$ is convex.
2. Show that a set K is convex if and only if $\lambda K + (1 - \lambda)K = K$ for every $\lambda \in [0, 1]$.
3. Let $f : \mathbb{R} \to \mathbb{R}$ be a convex function, and let $a < b$. Prove that f is Riemann integrable[1] on $[a, b]$.
4. Let $\varphi : \mathbb{R}^n \to (-\infty, \infty]$ be a convex function. Show that its sub-level sets $S_t := \{x \in \mathbb{R}^n : \varphi(x) \leq t\}$ are convex.
5. Let $f : \mathbb{R} \to \mathbb{R} \cup \{\infty\}$ be a convex function. Show that if f is closed then f is lower semicontinuous.
6. Let $f : \mathbb{R}^n \to \mathbb{R}$ be a convex function, and let x_0 be a point such that f is differentiable at x_0. Prove that $\partial f(x_0) = \nabla f(x_0)$.

Exercise 2

Let M_t be an $(\mathcal{F}_t)$-martingale. Prove that:

1. $\mathbb{E}[M_t] = \mathbb{E}[M_0]$, for all $t \geq 0$.
2. $\max(M_t, 0)$ is an $(\mathcal{F}_t)$-sub-martingale.
3. for any convex function φ such that $\varphi(M_t)$ is integrable for all t, $\varphi(M_t)$ is an $(\mathcal{F}_t)$-sub-martingale.

Exercise 3

Let $(X_i)_{i\in\mathbb{N}}$ be a sequence of bounded, independent random variables with $m_i := \mathbb{E}[X_i]$ and $\sigma_i^2 := \mathrm{Var}(X_i)$ for all $i \in \mathbb{N}$. Set $S_t = \sum_{i=1}^{t} X_i$ and $\mathcal{F}_t = \sigma(X_1, \ldots, X_t)$.

1. Find a sequence (a_t) of real numbers such that $S_t + a_t$ is an $\mathcal{F}_t$-martingale.
2. Find two sequences (a_t) and (b_t) of real numbers such that $S_t^2 + a_t S_t + b_t$ is an $\mathcal{F}_t$-martingale.
3. Fix $\lambda \in \mathbb{R}$. Find a sequence (a_t^λ) such that $(a_t^\lambda)^{-1} \exp(\lambda S_t)$ is an $\mathcal{F}_t$-martingale.

[1] The claim holds also for the Lebesgue measure.

Intermediate

Exercise 4

The goal of this exercise is to derive *Chernoff's inequality* and apply it (for more examples of concentration inequalities, see [3]).

Let us denote the moment generating function $M_X(t) = \mathbb{E}[e^{tX}]$.

1. Prove
$$\mathbb{P}(X \geq \alpha) \leq \inf_{t \geq 0} M_X(t) e^{-t\alpha}.$$

2. Compute the moment generating function of a Bernoulli random variable with parameter p.

3. Let X_i be independent Bernoulli random variables with parameter p_i. Consider their sum $S_N = \sum_{i=1}^{N} X_i$, and denotes its mean by $\mu = \mathbb{E}[S_N]$. Prove, for any $t > \mu$, we have
$$\mathbb{P}(S_N \geq t) \leq e^{-\mu} \left(\frac{e\mu}{t} \right)^t.$$

Exercise 5

Let $(X_n)_{n \in \mathbb{N}}$ be a stochastic process that starts at $X_0 = 1$, such that $X_n \in \{0, 2^n\}$ for all n, and is defined by the following operation:

$$\begin{cases} X_{n+1} = 0 & \text{if } X_n = 0 \\ X_{n+1} \sim \mathrm{Unif}\big(\{0, 2^n\}\big) & \text{if } X_n \neq 0 \end{cases}$$

1. Show that (X_n) is a martingale.

2. Show that (X_n) is L_1-bounded.

3. Show that (X_n) converges a.s. and determine the limit. Does it converge in L_1?

Exercise 6

The goal of this exercise is to derive some concentration inequalities on the hypercube[2].

Let $Q^n = \left[-\frac{1}{2}, \frac{1}{2}\right]^n$ be the unit cube.

1. Show that $\mathrm{diam}(Q^n) = \sqrt{n}$.

2. Let $X = (X_1, \ldots, X_n)$ and $Y = (Y_1, \ldots, Y_n)$ be two independent random vectors that are distributed uniformly on the cube. Show
$$\mathbb{E}[||X - Y||_2^2] = \frac{n}{6}.$$
What can you conclude?

3. Consider $\tilde{Q}^n = \sqrt{12} Q^n = [-\sqrt{3}, \sqrt{3}]^n$. Let Y be a random vector distributed on $\tilde{Q}^n$. Apply the central limit theorem on $\frac{1}{\sqrt{n}} \sum_i Y_i$.

4. Let $\theta_1, \ldots, \theta_n \in \mathbb{R}$ be some coefficients with $||\theta||_2 = 1$. Derive a concentration inequality for the random variable
$$\sum_{i=1}^{n} \theta_i Y_i.$$

Exercise 7

Let $K \subset \mathbb{R}^n$ be a convex set and suppose $\mathrm{Vol}_n(K) = 1$. Let $f : K \to \mathbb{R}$ be a C^1-smooth function with $\int_K f \mathrm{d}\lambda = 0$, where λ is the Lebesgue measure.

[2] In [2] you can find a similar derivation for the Euclidean ball as well.

1. Derive

$$\int_K f^2(x)\mathrm{d}\lambda(x) = \frac{1}{2}\int_K\int_K |f(x)-f(y)|^2\mathrm{d}x\mathrm{d}y\,.$$

2. Show that

$$f(x)-f(y) = \int_0^1 \nabla f\big((1-t)x+ty\big)(y-x)\mathrm{d}t\,.$$

3. Show that

$$\int_K f^2(x)\mathrm{d}\lambda(x) = \mathrm{diam}^2(K)\int_{\frac{1}{2}}^1\int_K\int_K \big|\nabla f\big((1-t)x+ty\big)\big|^2\mathrm{d}x\mathrm{d}y\mathrm{d}t\,,$$

4. Show that

$$\int_K \big|\nabla f\big((1-t)x+ty\big)\big|^2\mathrm{d}y \le 2^n\int_K |\nabla f(z)|^2\mathrm{d}z.$$

5. Conclude

$$\int_K f^2\mathrm{d}\lambda \le C_P(K)\int_K |\nabla f|^2\mathrm{d}\lambda,$$

   where $C_P(K) = 2^{n-1}\mathrm{diam}^2(K)$.

Exercise 8

1. Let $(X_t)_{t\in[0,\infty)}$ be a submartingale and $f:\mathbb{R}\to\mathbb{R}$ be an increasing convex function. Prove that, if $f(X_t)\in L^1$ for all $t\ge 0$, then $\big(f(X_t)\big)_{t\in[0,\infty)}$ is also a submartingale.
2. Prove that if $(X_t)_{t\in[0,\infty)}$ is a martingale and $f:\mathbb{R}\to\mathbb{R}$ is convex (but not necessarily increasing), then, if $\big(f(X_t)\big)_{t\in[0,\infty)}\in L^1$ for all $t\ge 0$, we have that $\big(f(X_t)\big)_{t\in[0,\infty)}$ is a submartingale.
3. Give an example of a martingale $(X_t)_{t\in[0,\infty)}$ and a convex $f:\mathbb{R}\to\mathbb{R}$ such that $\big(f(X_t)\big)_{t\in[0,\infty)}$ is a submartingale but not a martingale.

Advanced

Exercise 9

The following theorem (see [3]) proves that dense graphs are almost regular.
Prove there is an absolute constant C such that the following holds: consider a random graph $G\sim G(n,p)$ with expected degree satisfying $d\ge C\log n$. Then, with probability at least 0.9, the following occurs: all vertices of G have degrees between $0.9d$ and $1.1d$.

Exercise 10

The following inequality is known as *Bennett's inequality.* (see [3] for a more general statement).
Let $X_1,\ldots,X_N$ be i.i.d. random variables. Assume that $\mathbb{E}[X_i]=0$, $\mathrm{Var}(X_i)=\sigma^2$ and $|X_i|\le M\ \forall i$.
Prove, for any $t>0$, that we have

$$\mathbb{P}\left(\sum_{i=1}^N X_i\ge t\right)\le\exp\left(-\frac{\sigma^2}{K^2}h\left(\frac{Kt}{\sigma^2}\right)\right),$$

where $h(u)=(1+u)\log(1+u)-u$.

Exercise 11
Let $f : [0,1] \to \mathbb{R}$ be a Lipschitz function. Prove there $\exists g$ measurable and bounded such that $\forall x \in \mathbb{R}$ it holds

$$f(x) = f(0) + \int_0^x g(t)\mathrm{d}t\,.$$

Hint [1]: let $X \sim$ *Unif*$[0,1]$. *Find two suitable sequences of r.v.* (X_n) *and* (Z_n) *depending on* X.

References

[1] Max Fathi. *Probabilités M1*. 2023.

[2] Bo'az Klartag. *Regularity through convexity in high dimensions*. 2014.

[3] Roman Vershynin. *High-Dimensional Probability. An Introduction with Applications in Data Science*. Cambridge University Press, 2018.