Exercise Sheet 3

Exercise 1

* Let $x, y \in \Omega$. $Q(x,y) = \underbrace{\frac{P(x,y)}{2}}_{\geq 0} + \underbrace{\frac{I(x,y)}{2}}_{\geq 0} \geq 0$

* Let $x \in \Omega$.

$$\sum_{y\in\Omega} Q(x,y) = \frac{1}{2} \underbrace{\sum_{y\in\Omega} P(x,y)}_{=1} + \frac{1}{2} \underbrace{\sum_{y\in\Omega} I(x,y)}_{=1} = 1$$

* Let $x, y \in \Omega$

$$\sum_{x\in\Omega} \pi(x)\, Q(x,y) = \frac{1}{2}\sum_{x\in\Omega} \pi(x) P(x,y) + \frac{1}{2}\sum_{x\in\Omega} \pi(x) I(x,y) = \pi(y) \qquad \square$$

Exercise 2

Let $B = \{x : \mu(x) \geq \nu(x)\}$, let $A \subset \Omega$.

Then $\mu(A) - \nu(A) \leq \mu(A\cap B) - \nu(A \cap B) \leq \mu(B) - \nu(B)$

$\quad$ if $x \in A \cap B^c \Rightarrow \mu(x) - \nu(x) < 0$

Similarly, $\nu(A) - \mu(A) \leq \nu(B^c) - \mu(B^c)$

$$\|\mu - \nu\|_{TV} = \frac{1}{2}\left| \mu(B) - \nu(B) + \nu(B^c) - \mu(B^c) \right| = \frac{1}{2}\sum_{x\in\Omega} |\mu(x) - \nu(x)| \qquad \square$$

Exercise 3

$$\langle f, Pg\rangle_\pi = \sum_{x,y\in\Omega} \pi(x) f(x) P(x,y) g(y)$$
$$= \sum_{x,y\in\Omega} \pi(y) f(x) P^*(y,x) g(y)$$
$$= \langle P^* f, g\rangle \qquad \square$$

Exercise 4

$$\mathcal{E}_P(f,f) = \langle f(I-P), f\rangle_\pi = \langle f, f\rangle_\pi - \langle f, Pf\rangle_\pi$$
$$= \langle f, f\rangle_\pi - \langle P^* f, f\rangle_\pi = \langle f(I-P^*), f\rangle_\pi = \mathcal{E}_{P^*}(f,f)$$

$$\mathcal{E}_{\frac{P+P^*}{2}} = \langle f\left(I - \frac{P+P^*}{2}\right), f\rangle_\pi$$
$$= \langle f, f\rangle_\pi - \langle f\frac{P}{2}, f\rangle_\pi - \langle f\frac{P^*}{2}, f\rangle_\pi$$
$$= \langle f, f\rangle_\pi - \langle fP, f\rangle_\pi = \mathcal{E}_P(f,f) \qquad \square$$

Exercise 5

$$\mathcal{E}_P(f,g) = \sum_{x,y\in\Omega} \pi(x) f(x) P(x,y) g(y)$$
$$= \sum_{x,y\in\Omega} \pi(y) f(x) P(y,x) g(y)$$
$$= \mathcal{E}_P(g,f) \qquad \square$$

Exercise 6

$$\left\| \frac{\mu}{\pi} - 1 \right\|_{1,\pi} = \sum_{y\in\Omega} |k_n(y) - 1| \, \pi(y)$$
$$= \sum_{y\in\Omega} \left| \frac{\mu(y)}{\pi(y)} - 1 \right| \pi(y)$$
$$= \sum_{y\in\Omega} |\mu(y) - \pi(y)| = 2\|\mu - \nu\|_{TV}.$$

$$\left\| \frac{\mu}{\pi} - 1 \right\|_{2,\pi} = \sum_{y\in\Omega} |k_n(y) - 1|^2 \pi(y) = \sum_{y\in\Omega} \left| \frac{\mu(y)}{\pi(y)} - 1 \right|^2 \pi(y) = \mathrm{var}_\pi\left(\frac{\mu}{\pi}\right)$$

Exercise 7

Let $d\in[0,1]$.

$D(\mu\|\pi) = \mathrm{Ent}_\pi\left(\frac{\mu}{\pi}\right) = E_\pi\left(\frac{\mu}{\pi}\right)\log\left(\frac{\mu}{\pi}\right)$ is convex because $f\log f$ is convex.

Exercise 8

$$\begin{aligned} d(\sigma P^n, \pi) &= d\Big(\sum_{x\in\Omega} \sigma(x) P^n(x,\cdot), \pi\Big) \\ &\le \sum_{x\in\Omega} \sigma(x)\, d(P^n(x,\cdot), \pi) \\ &\le \max_{x\in\Omega} d(P^n(x,\cdot), \pi) \end{aligned}$$

$\to$ distance is maximized when the initial distribution is concentrated at a point.

Exercise 9

$$\frac{d}{dt} h_t = \frac{d}{dt} H_t^* h_0 = \frac{d}{dt} e^{t\mathcal{L}^*} h_0 = \mathcal{L}^* e^{t\mathcal{L}^*} h_0 = \mathcal{L}^* h_t$$

Exercise 10

$$\begin{aligned} \frac{d}{dt} \mathrm{var}(h_t) &= \int \frac{d}{dt} h_t^2 \, d\pi = 2 \int h_t \mathcal{L}^* h_t \, d\pi \\ &= 2 \int \mathcal{L}(h_t) h_t \, d\pi = -2\, \mathcal{E}(h_t, h_t) \end{aligned}$$

$$\begin{aligned} \frac{d}{dt} \mathrm{Ent}(h_t) &= \int \frac{d}{dt} h_t \log h_t \, d\pi = \int (\log h_t + 1) \mathcal{L}^* h_t \, d\pi \\ &= \int \mathcal{L}(\log h_t) h_t \, d\pi = -\mathcal{E}(h_t, \log h_t) \end{aligned}$$

Exercise 11

We have the following differential equation

$$\frac{d}{dt} \operatorname{var}(h_t^x) = -2\,\mathcal{E}(h_t^x, h_t^x)$$
$$\leq -2\lambda \operatorname{var}(h_t^x)$$

with $\operatorname{var}(h_0) \leq \frac{1-\pi^*}{\pi^*}$

$$\Rightarrow \begin{cases} y'(t) = -2\lambda\, y(t) \\ y(0) = \frac{1-\pi^*}{\pi^*} \end{cases} \Rightarrow \quad y(t) = \frac{1-\pi^*}{\pi^*} e^{-2\lambda t}$$

Thus $\operatorname{var}(h_t) \leq \frac{1-\pi^*}{\pi^*} e^{-2\lambda t}$

We need to solve $\frac{1-\pi^*}{\pi^*} e^{-2\lambda t} \overset{!}{<} \varepsilon$ and we get the claim.

Exercise 12

We have the following differential equation

$$\frac{d}{dt} \operatorname{Ent}(h_t^x) = -\mathcal{E}(h_t^x, \log h_t^x)$$
$$\leq -\rho_0 \operatorname{Ent}(h_t^x).$$

Exercise 13

Since $k_{i+1} - 1 = P^*(k_i - 1)$

$\mathbb{E}[k_i - 1] = 0$ for all i

$$\Rightarrow \| k_n - 1 \|_2 = \| P^{*n}(k_0 - 1) \|_2 \leq \| P^* \|^n \, \| k_0 - 1 \|_2 \qquad (\ast\ast)$$

Solve $\|k_n - 1\|_2 \leq \|P^*\|^n \|k_0 - 1\|$

$$\leq \|P^*\|^n \sqrt{\frac{1-\pi^*}{\pi^*}} < \varepsilon$$

and use the fact that $\log x \leq -(1-x)$ and $\|k_0 - 1\|_2 \leq \sqrt{\frac{1-\pi^*}{\pi^*}}$.

Exercise 14

① Note that $\mathbb{E}_\pi[f] = \mathbb{E}_\pi[Kf]$ for any transition probability matrix K

$(\pi(y) = \sum_x \pi(x) K(x,y))$

$$\Rightarrow \operatorname{var}(P^* f) - \operatorname{var}(f) = \langle P^* f, P^* f \rangle_\pi - \langle f, f \rangle_\pi$$

$$= -\langle f, (I - PP^*) f \rangle_\pi$$

② $\operatorname{var}(k_n) \leq \operatorname{var}(k_{n-1})(1 - d_{PP^*})$ by ①

$\overset{\text{induction}}{\Longrightarrow} \operatorname{var}(k_n) \leq \operatorname{var}(k_0)(1 - d_{PP^*})^n$

Solve $\operatorname{var}(k_0)(1 - d_{PP^*})^n < \varepsilon^2$ and use $\log(1 - d_{PP^*}) \leq -d_{PP}$.

Exercise 15

$\forall x \in \Omega : \quad P^*(x,x) = P(x,x) \geq \alpha$

$$\Rightarrow \pi(x) PP^*(x,y) \geq \pi(x) P(x,x) P^*(x,y) + \pi(x) P(x,y) P^*(y,y)$$

$$\geq \alpha \pi(y) P(y,x) + \alpha \pi(x) P(x,y)$$

$$\Rightarrow \mathcal{E}_{PP^*}(f,f) \geq \alpha \mathcal{E}(f,f) + \alpha \mathcal{E}(f,f) = 2\alpha \mathcal{E}(f,f)$$

$$\Rightarrow d_{PP^*} \geq 2\alpha d$$

From Exercise 14 it follows $\tau_2(\varepsilon) \leq \left\lceil \frac{1}{\alpha d} \log \frac{1}{\varepsilon \sqrt{\pi_*}} \right\rceil$

Exercise 16

$G(r) = dr$

$$\Rightarrow T_2(\varepsilon) \leq \int_{var(P_0)}^{\varepsilon^2} \frac{dI}{-2G(I)} = \int_{var(P_0)}^{\varepsilon^2} \frac{dr}{-2dr} = -\frac{1}{2d}\log\left(\frac{\varepsilon^2}{var(P_0)}\right)$$

$$= -\frac{1}{2d}\log(\varepsilon^2) + \frac{1}{2d}\log var(P_0)$$

$$\leq -\frac{1}{2d}\log(\varepsilon^2) + \frac{1}{2d}\log\frac{1-\pi^*}{\pi^*}$$

$\rightarrow$ exercise 11

Exercise 17

By definition

$$Ent(f^2) = \mathbb{E}\, f^2 \log\frac{f^2}{\mathbb{E}f^2} = \underbrace{2\mathbb{E}f^2 \log\frac{f\,\mathbb{E}f}{\mathbb{E}f^2}}_{A} + \mathbb{E}f^2 \log\frac{\mathbb{E}f^2}{(\mathbb{E}f)^2}$$

We show that $A \geq 0$ and drops out.

Use the approximation $\log x \geq 1 - \frac{1}{x}$

$$\mathbb{E}\, f^2 \log\frac{f\,\mathbb{E}f}{\mathbb{E}f^2} \geq \mathbb{E}f^2\left(1 - \frac{\mathbb{E}f^2}{f\,\mathbb{E}f}\right) = \mathbb{E}f^2 - \mathbb{E}f^2\frac{\mathbb{E}f}{\mathbb{E}f} = 0.$$

If $\mathbb{E}[f] = 1$

$$\mathcal{E}(f,f) \geq \rho\, Ent(f^2) \geq \rho\,(1+var(f))\log(1+var(f))$$

Exercise 18

Apply equation (11) and exercise 17

$$T_2(\varepsilon) \le \int_{\mathrm{var}(h_0)}^{4} \frac{dI}{-2\rho(1+I)\log(1+I)} + \int_{4}^{\varepsilon^2} \frac{dI}{-2\alpha I}$$

$$= \frac{-1}{2\rho}\left(\log\log(1+4) - \log\log\big(1+\underbrace{\mathrm{var}(h_0)}_{\le \frac{1-\pi^*}{\pi^*}}\big)\right) - \frac{1}{2\alpha}\log\frac{\varepsilon^2}{4}$$

apply $\rho \le \frac{\alpha}{2}$ to $\log\log s$ term.

Exercise 19

Nash inequality can be rewritten as

$$\mathcal{E}(f,f) \geq \|f\|_2^2 \left(\frac{1}{C}\left(\frac{\|f\|_2}{\|f\|_1}\right)^{1/D} - \frac{1}{T}\right)$$

use the facts that $\|f\|_1 = \mathbb{E} f = 1$

$$\mathrm{var}(f) = \|f\|_2^2 - 1 .$$

Exercise 20

Use Exercise 19 when $\mathrm{var}(h_t) \geq \left(\frac{DC}{T}\right)^D - 1$ Ⓐ

spectral bound $\mathrm{var}(h_t) \le \left(\frac{DC}{T}\right)^D - 1$ Ⓑ

Ⓐ $$\int_{\mathrm{var}(h_0)}^{\left(\frac{DC}{T}\right)^D - 1} \frac{dI}{-2(1+I)\left(\frac{(1+I)^{1/D}}{C} - \frac{1}{T}\right)} = -\frac{DT}{2}\log\left(1 - \frac{C/T}{(1+I)^{1/D}}\right)\Bigg|_{\mathrm{var}(h_0)}^{\left(\frac{DC}{T}\right)^D - 1}$$

$$\le -\frac{DT}{2}\log\left(1 - \frac{1}{D}\right) \le \frac{DT}{2(D-1)} \le T$$

$$\log\left(1 - \frac{1}{x}\right) \geq -\frac{1}{x-1}$$

for Ⓑ $\to$ same as in Exercise 18.