Exercise Sheet 5
Brownian Motion and Stochastic Calculus

Exercise 1

Let $(B_t)_{t\geq 0}$ be a standard Brownian motion.

1. For all $h > 0$ prove that $(B_{t+h} - B_h)_{t\geq 0}$ is a Brownian motion.
2. For any $\alpha > 0$, prove that the process $\left(\frac{1}{\sqrt{\alpha}} B_{\alpha t}\right)_{t\geq 0}$ is also a Brownian motion.
3. Let W be a Brownian motion on the filtered probability space $(\Omega, \mathcal{F}, \mathbb{F}, P)$, whith $\mathbb{F} = \{\mathcal{F}_t : t \geq 0\}$. Let $X_t = \exp\left(\sigma W_t - \frac{\sigma^2}{2} t\right)$. Show that $X = \{X_t : t \geq 0\}$ is a martingale.

Exercise 2

Prove almost surely that

$$\lim_{t\to\infty} \frac{B_t}{t} = 0$$

and that the process

$$X_t = \begin{cases} tB_{\frac{1}{t}} & t > 0 \\ 0 & t = 0 \end{cases}$$

is a Brownian motion.

Exercise 3

Give the semimartingale decomposition of the following processes

1. $X_t = t^2 B_t^5$.
2. $Y_t = \exp(tB_t)$.
3. $Z_t = B_t^3 - 3tB_t$.

Exercise 4

Let $b \in \mathbb{R}, \sigma > 0$ and B the standard one-dimensional Brownian motion. S is called *geometric Brownian motion* if it is the solution to the stochastic differential equation

$$\mathrm{d}S_t = bS_t\mathrm{d}t + \sigma S_t \mathrm{d}B_t, \qquad S_0 = 1\,. \tag{1}$$

1. Show that $Y_t = e^{-bt} S_t$ solves
$$\mathrm{d}Y_t = \sigma Y_t \mathrm{d}B_t\,.$$
2. Show that
$$S_t = e^{bt} \exp\left(\sigma B_t - \frac{\sigma^2 t}{2}\right)\,.$$

Exercise 5

When f is a smooth function with $f(0) = 0$, we have

$$\int_0^t \operatorname{sgn}\big(f(s))\big)\mathrm{d}f(s) = |f(s)|\,,$$

where $\operatorname{sgn}(x) = \mathbb{1}_{\{x\geq 0\}} - \mathbb{1}_{\{x<0\}}$. The goal of this exercise is to show how different things are when one replaces f by Brownian motion. this is called *Tanaka's example.*

1. Show that if X is a one-dimensional Brownian motion, then
$$B_t := \int_0^t \operatorname{sgn}(X_s)\mathrm{d}X_s$$
is a Brownian motion.
2. Show that
$$X_t = \int_0^t \operatorname{sgn}(X_s)\mathrm{d}B_s\,.$$
3. Show that if $Y = -X$, then
$$Y_t = \int_0^t \operatorname{sgn}(Y_s)\mathrm{d}B_s\,.$$

Thus, we have seen that the SDE doesn't have a unique solution.

Exercise 6

Doob's maximal inequality states that for any non-negative sub-martingale with almost-surely right-continuous sample paths, we have for any $r > 0$
$$\mathbb{P}\left(\sup_{0\le t\le T} X_t \ge r\right) \le \frac{\mathbb{E}[X_t]}{r}.$$
Use this inequality to show that for a Brownian motion starting from zero, we have
$$\mathbb{P}\left(\sup_{0\le t\le T} B_t \ge r\right) \le \exp\left(-\frac{r^2}{2T}\right).$$

Advanced

Exercise 7

Let $\sigma : \mathbb{R} \to (0,\infty)$ be a continuous function, not necessarily bounded. The goal of the exercise is to show that any (weak) solution to the one-dimensional stochastic differential equation
$$\mathrm{d}X_t = \sigma(X_t)\mathrm{d}B_t, \qquad X_0 = 0 \tag{2}$$
cannot explode, i.e. $T = \infty$ where T be the explosion time of the solution (here it means that X is defined for all $t \ge 0$).

1. Show that there exists a Brownian motion β such that $X_t = \beta_{\langle X\rangle_t}$ and that $\langle X\rangle_t \to \infty$ as $t \to \infty$.
2. Using $\mathrm{d}B_t = \frac{1}{\sigma(X_t)}\mathrm{d}X_t$ and the fact that the one-dimensional Brownian motion is recurrent, show that $T = \infty$ a.s.