Exercise Sheet 5.1

Exercise 1

1. Let $Z_t = B_{t+h} - B_h$

* The process starts at 0: $Z_0 = B_h - B_h = 0$

* Independent stationary increments: $\forall\, t \geq s \geq 0$

$Z_t - Z_s = B_{t+h} - B_h - B_{s+h} + B_h = B_{t+h} - B_{s+h} \longrightarrow$ these are $\perp\!\!\!\perp$ stationary increments

because $(B_t)_{t\geq 0}$ is a BM.

* Normally distributed increments $\forall s,t \geq 0$, $s<t$

$\mathbb{E}[Z_t - Z_s] = \mathbb{E}[B_{t+h} - B_{s+h}] = 0$

$\operatorname{var}(Z_t - Z_s) = \operatorname{var}(B_{t+h} - B_{s+h}) = (t+h) - (s+h) = t-s$

* Continuity

$\forall \omega \in \Omega$, the path $t \mapsto Z_t(\omega) = B_{t+h}(\omega) - B_h(\omega)$ is continuous since

$t \mapsto B_t(\omega)$ is continuous.

2. * The process starts at 0: $\frac{1}{\sqrt{\alpha}} B_{\alpha \cdot 0} = \frac{1}{\sqrt{\alpha}} B_0 = 0$

* Independent stationary increments $\forall t,s > 0$

$\frac{1}{\sqrt{\alpha}} B_{\alpha t} - \frac{1}{\sqrt{\alpha}} B_{\alpha s} = \frac{1}{\sqrt{\alpha}} (B_{\alpha t} - B_{\alpha s})$

since the increments of a BM are $\perp\!\!\!\perp$ and stationary

$\Rightarrow \frac{1}{\sqrt{\alpha}} B_{\alpha t}$ are also $\perp\!\!\!\perp$ and stationary

* Normally distributed increments

$$\mathbb{E}\left[\frac{1}{\sqrt{\alpha}} B_{\alpha t} - \frac{1}{\sqrt{\alpha}} B_{\alpha s}\right] = \mathbb{E}\left[\frac{1}{\sqrt{\alpha}} B_{\alpha(t-s)}\right] = 0$$

$$\mathrm{var}\left(\frac{1}{\sqrt{\alpha}} B_{\alpha t} - \frac{1}{\sqrt{\alpha}} B_{\alpha s}\right) = \mathrm{var}\left(\frac{1}{\sqrt{\alpha}} B_{\alpha(t-s)}\right) = \alpha(t-s)$$

* Continuity: same as 1

3 * Integrability

$$\mathbb{E}[|X_t|] = \mathbb{E}\left[\left|\exp\left(\sigma W_t - \frac{\sigma^2}{2} t\right)\right|\right]$$

$$= \mathbb{E}\left[\exp\left(\sigma W_t - \frac{\sigma^2}{2} t\right)\right]$$

$$= \int_{\mathbb{R}} \frac{1}{\sqrt{2\pi t}} \exp\left(-\frac{w^2 - 2t\sigma w + \sigma^2 t^2}{2t}\right) dw$$

$$= \underbrace{\int_{\mathbb{R}} \frac{1}{\sqrt{2\pi t}} \exp\left(-\frac{(w-t\sigma)^2}{2t}\right) dw}_{\mathcal{N}(t\sigma,\, t) \text{ density function}} = 1 < \infty$$

* Since X_t is a continuous function of $\mathcal{F}_t$ - measurable r.v., X_t is $\mathcal{F}_t$ - measurable.

* $\forall\, 0 \leq s \leq t$

$$\mathbb{E}[X_t \mid \mathcal{F}_s] = X_s\, \mathbb{E}\left[\frac{X_t}{X_s} \,\middle|\, \mathcal{F}_s\right] \qquad \text{since } X_s > 0$$

$$= X_s\, \mathbb{E}\left[\frac{\exp\left(\sigma W_t - \frac{\sigma^2}{2} t\right)}{\exp\left(\sigma W_s - \frac{\sigma^2}{2} s\right)} \,\middle|\, \mathcal{F}_s\right]$$

$$= X_s\, \mathbb{E}\left[\exp\left(\sigma(W_t - W_s) - \frac{\sigma^2}{2}(t-s)\right) \,\middle|\, \mathcal{F}_s\right]$$

$$= X_s\, \mathbb{E}\left[\exp\left(\sigma(W_t - W_s) - \frac{\sigma^2}{2}(t-s)\right)\right] \qquad \text{since } W_t - W_s \perp \mathcal{F}_s$$

$$= X_s \int_{\mathbb{R}} \exp\left(\sigma w - \frac{\sigma^2}{2}(t-s)\right) \frac{1}{\sqrt{2\pi}} \frac{1}{\sqrt{t-s}} \exp\left(-\frac{w^2}{2(t-s)}\right) dw$$

$$= X_s \int_{\mathbb{R}} \frac{1}{\sqrt{2\pi}} \frac{1}{\sqrt{t-s}} \exp\left(-\frac{w^2 - 2(t-s)\sigma w + \sigma^2 (t-s)^2}{2(t-s)}\right) dw$$

$$= X_s \underbrace{\int_{\mathbb{R}} \frac{1}{\sqrt{2\pi}} \frac{1}{\sqrt{t-s}} \exp\left(-\frac{(w-(t-s)\sigma)^2}{2(t-s)}\right) dw}_{=1 \text{ since this is } \mathcal{N}((t-s)\sigma,\, t-s) \text{ density}} = X_s$$

Exercise 2

* Let $X_t^\sigma = \exp\left(\sigma B_t - \frac{\sigma^2}{2} t\right)$.

If $\sigma > 0$ then $\limsup_{t\to\infty} \frac{B_t}{t} > \sigma \Rightarrow \limsup_{t\to\infty} X_t^\sigma = \infty$

Since X_t^σ is a martingale, by Doob's martingale thm X_t^σ converges a.s.

$\Rightarrow \mathbb{P}\left(\limsup_{t\to\infty} \frac{B_t}{t} > \sigma\right) = 0$. Let $\sigma \downarrow 0$, $\limsup_{t\to\infty} \frac{B_t}{t} \leq 0$.

Since the law of B_t is symmetric, $\liminf_{t\to\infty} \frac{B_t}{t} \geq 0$ a.s.

* $X_0 = 0$ ✓

* $\forall s,t \geq 0$ st $s<t$

$$X_t - X_s = t B_{\frac{1}{t}} - s B_{\frac{1}{s}}$$

$$= -s\left(B_{\frac{1}{s}} - B_{\frac{1}{t}}\right) + (t-s) B_{\frac{1}{t}}$$

is a linear combination of two $\perp\!\!\!\perp$ Gaussian r.v.

$\Rightarrow$ $X_t - X_u$ is Gaussian with

$$\mathbb{E}[X_t - X_s] = \mathbb{E}\left[t B_{\frac{1}{t}} - s B_{\frac{1}{s}}\right] = t\,\mathbb{E}\left[B_{\frac{1}{t}}\right] - s\,\mathbb{E}\left[B_{\frac{1}{s}}\right] = 0$$

$$\operatorname{var}(X_t - X_s) = \operatorname{var}\left(-s\left(B_{\frac{1}{s}} - B_{\frac{1}{t}}\right) + (t-s) B_{\frac{1}{t}}\right)$$

$$= \operatorname{var}\left(-s\left(B_{\frac{1}{s}} - B_{\frac{1}{t}}\right)\right) + \operatorname{var}\left((t-s) B_{\frac{1}{t}}\right)$$

$$= s^2 \operatorname{var}\left(B_{\frac{1}{s}} - B_{\frac{1}{t}}\right) + (t-s)^2 \operatorname{var}\left(B_{\frac{1}{t}}\right)$$

$$= s^2\left(\frac{1}{s} - \frac{1}{t}\right) + (t-s)^2 \frac{1}{t}$$

$$= t-s.$$

$(dB_t)^2 = dt$

$dB_t\, dt = 0$

$(dt)^2 = 0$

<u>Ex</u> $dX_t = m\,dt + s\,dB_t$

$$(dX_t)^2 = (m\,dt + s\,dB_t)^2$$

$$= (m\,dt)^2 + (s\,dB_t)^2 + 2(m\,dt)(s\,dB_t)$$

$$= 0 + s^2 dt + 0 = s^2 dt$$

$$(dX_t)^3 = (dX_t)(dX_t)^2 = (m\,dt + s\,dB_t)\, s^2 dt$$

$$= m s^2 (dt)^2 + s\,dB_t\, dt = 0$$

Exercise 3

① $X_t = t^2 B_t^5$

Let $f(x,y) = x^2 y^5$

$$X_t = f(t, B_t)$$

$$= X_0 + \int_0^t \partial_x f(s,B_s)\, ds + \int_0^t \partial_y f(s, B_s)\, dB_s + \frac{1}{2}\int_0^t \partial_{yy} f(s, B_s)\, ds$$

$$= \underbrace{\int_0^t 5s^2 B_s^4 \, dB_s}_{\text{local martingale}} + \underbrace{\int_0^t (2s B_s^5 + 10 s^2 B_s^3 \, ds)}_{\text{bounded variation process}}$$

② $Y_t = \exp(t B_t)$

$f(x,y) = \exp(xy)$

$\partial_x f = y e^{xy}$ $\qquad$ $\partial_y = x e^{xy}$

$\partial_{yy} f = x^2 e^{xy}$

$$\Rightarrow Y_t = 1 + \int_0^t s\exp(sB_s)\, dB_s + \int_0^t B_s \exp(sB_s) ds + \frac{1}{2}\int_0^t s^2 \exp(sB_s)\, ds$$

③ $f(x,y) = y^3 - 3xy$

$\partial_x f = -3y$ $\qquad$ $\partial_y f = 3y^2 - 3x$

$\partial_{yy} f = 6y$

Note that $\partial_x f + \frac{1}{2}\partial_{yy} f = 0 \Rightarrow Z_t$ is a local martingale

$$Z_t = \int_0^t \partial_y f(s, B_s)\, dB_s = \int_0^t 3B_s^2 - 3s dB_s$$

Exercise 4

① $x \mapsto bx$ and $x \mapsto \sigma x$ are globally Lipschitz $\Rightarrow$ the strong solution to (1) exists $\forall t \geq 0$ and is unique up to 0-probability event.

Itô : $\quad Y_t = f(t, S_t) \quad$ where $\quad f(x,y) = e^{-bx} y$

$$dY_t = e^{-bt} dS_t - be^{-bt} S_t \, dt$$

$$= e^{-bt} (b S_t \, dt + \sigma S_t \, dB_t) - b e^{-bt} S_t \, dt$$

$$= \sigma Y_t \, dB_t$$

② The SDE for Y has also a unique strong solution given that $Y_0 = 1$.

This equation characterizes the exponential martingale $\mathcal{E}(\sigma B)$ which gives

$$Y_t = \exp\left(\sigma B_t - \frac{\sigma^2 t}{2}\right)$$

$$\Rightarrow \quad S_t = e^{bt} S_t = \exp(B_t) \exp\left(\sigma B_t - \frac{\sigma^2 t}{2}\right)$$

Exercise 5

1. The process B_t defined as $\int_0^t sgn(X_s)\, dX_s$ is a local martingale

$\Rightarrow \quad \langle B_t \rangle = \int_0^t sgn(X_s)^2 \, d\langle X \rangle_s = \int_0^t 1\, ds = t$

$\Rightarrow$ B is a BM from Lévy's characterization thm.

2. $sgn(X) \cdot B = sgn(X) \cdot (sgn(X) \cdot X) = (sgn(X))^2 \cdot X = X$

$\rightarrow$ a.s. $\forall t \geq 0 \quad X_t = \int_0^t sgn(X_s)\, dB_s$

3. If $Y = -X \Rightarrow Y = -sgn(X) \cdot B = sgn(Y) \cdot B - 2\, \mathbb{1}_{\{X=0\}} \cdot B$

The local martingale

$$Z_t := \mathbb{1}(X=0 \cdot B)_t = \int_0^t \mathbb{1}_{\{X_s=0\}}\, dB_s$$

has quadratic variation $\langle Z \rangle_t = \int_0^t \mathbb{1}_{X_s=0}\, ds$

$\mathbb{E}[\langle Z \rangle_t] = \mathbb{E}\left[\int_0^t \mathbb{1}_{\{X_s=0\}}\, ds\right] = \int_0^t \mathbb{P}(X_s=0)\, ds = 0$

therefore $\langle Z \rangle$ is a.s. constantly 0 which implies $Z=0$

We conclude $Y = sgn(Y) \cdot B$.

Exercise 6

$$\left\{ \sup_{0\le t\le T} B_t \geq C \right\} = \left\{ \sup_{0\le t\le T} \exp(\alpha B_t) \geq \exp(\alpha C) \right\}$$

$$\begin{aligned}
\mathbb{P}\left(\sup_{0\le t\le T} B_t \geq C \right) &= \mathbb{P}\left(\sup_{0\le t\le T} \exp(\alpha B_t) \geq \exp(\alpha C) \right) \\
&\leq \frac{\mathbb{E}[\exp(\alpha B_T)]}{\exp(\alpha C)} \\
&= \exp\left(\tfrac{1}{2}\alpha^2 T - \alpha C \right)
\end{aligned}$$

minimize over α, $\alpha_{opt} = \frac{C}{T}$.

Exercise 7

(1) If (X, B) is a solution to (2) $\Rightarrow$ X is a local martingale

Then $\exists$ BM β, $\{X_t,\ t< T\} = \{\beta_{\langle X\rangle_t},\ t< T\}$ is unbounded

therefore $\langle X\rangle_t \to \infty$ as $t\to\infty$.

(2) Since $\sigma(x)>0$ $\forall x\in\mathbb{R}$ and $dB_t = \frac{1}{\sigma(X_t)}\, dX_t$

$$t = \langle B\rangle_t = \int_0^t \frac{1}{\sigma(X_s)^2}\, d\langle X\rangle_s = \int_0^t \frac{1}{\sigma(\beta_{\langle X\rangle_s})^2}\, d\langle X\rangle_s = \int_0^{\langle X\rangle_t} \frac{1}{\sigma(\beta_r)^2}\, dr \qquad (\star)$$

From the recurrence of one-dim BM (that β visits every real number ∞-many times),

the RHS of $(\star)$ $\to\infty$ as $\langle X\rangle_t \to \langle X\rangle_T = \infty$ a.s.

Therefore, $T=\infty$ a.s.