Exercise Sheet 7
Stochastic Localization

Let μ be an absolute continuous probability measure on $\mathbb{R}^n$ with density ν_0 and with finite second moment. Let $(B_t)_{t\geq 0}$ be a standard Brownian Motion on $\mathbb{R}^n$ with $B_0 = 0$. We know that stochastic localization is a stochastic process $(\nu_t(x))_{t\geq 0}$, with $\frac{\mathrm{d}\nu_t}{\mathrm{d}\mu}(x) = F_t(x)$ which is the solution of the stochastic differential equation

$$\begin{cases} \mathrm{d}F_t(x) = F_t(x)\langle C_t(x - a_t), \mathrm{d}B_t\rangle \quad \forall x \in \mathbb{R}^n\,, \\ F_0(x) = 1 \end{cases} \tag{1}$$

where C_t is an adapted process w.r.t B_t and

$$a_t = \int_{\mathbb{R}^n} x\nu_t(x)\mathrm{d}x\,.$$

We recall the following important properties.

(A) Almost surely, for all t, ν_t is a probability measure.

(B) For all $A \subset \mathbb{R}^n$, $\nu_t(A)$ is a martingale.

(C) The process a_t converges to a point $a_\infty := \lim_{t\to\infty} a_t \in \mathbb{R}^n$, which is distributed according to the law μ. Moreover, the measure ν_t almost surely weakly converges to a Dirac measure at a_∞.

Exercise 1

Let $\varphi : \mathbb{R}^n \to \mathbb{R}$ be a continuous and bounded function. Then $\int_{\mathbb{R}^n} \varphi(x)\mathrm{d}\nu_t(x)$ is a martingale. Conclude that for any test function $\varphi : \mathbb{R}^n \to \mathbb{R}$ we have

$$\mathbb{E}_{X\sim\nu_0}[\varphi(X)] = \mathbb{E}\left[\mathbb{E}_{X\sim\nu_t}[\varphi(X)]\right]\,, \tag{2}$$

where the outer expectation is over the randomness of the process.

We now give another interpretation of (1), which can be described as a **stochastic tilt** of ν_0. For simplicity let us assume $C_t = \mathrm{Id}$. For $t \geq 0$ and $\theta \in \mathbb{R}^n$, let $\nu_{t,\theta}$ denote the probability density given by

$$\nu_{t,\theta}(x) = \frac{1}{Z(t,\theta)} \exp\left(\langle\theta, x\rangle - \frac{t||x||_2^2}{2}\right)\nu_0(x)\,,$$

where

$$Z(t,\theta) = \int_{\mathbb{R}^n} \exp\left(\langle\theta, x\rangle - \frac{t||x||_2^2}{2}\right)\nu_0(x)\mathrm{d}x$$

is the normalization constant. Let $a(t,\theta) = \int_{\mathbb{R}^n} x\nu_{t,\theta}(x)\mathrm{d}x$ and define

$$\begin{cases} \mathrm{d}\theta_t = a(t,\theta_t)\mathrm{d}t + \mathrm{d}B_t \\ \theta_0 = 0 \end{cases} \tag{3}$$

Exercise 2

Consider again (1) with $C_t = \mathrm{Id}$.

1. Compute $\mathrm{d}\log F_t(x)$.

2. Show that it holds
$$F_t(x) = \frac{1}{Z_t} \exp\left(\langle \theta_t, x \rangle - \frac{1}{2} t ||x||_2^2 \right) . \tag{4}$$

Let $E \subset \mathbb{R}^n$. Define
$$M_t = \int_E \mathrm{d}\nu_t(x)$$
Let A_t be the covariance matrix of ν_t, namely
$$A_t = \mathbb{E}_{X \sim \nu_t}[(X - a_t)(X - a_t)^T] .$$

Exercise 3

Let us suppose $C_t = \mathrm{Id}$.

1. Prove that
$$\mathrm{d}M_t = \int_E \langle x - a_t, \mathrm{d}B_t \rangle \mathrm{d}\mu(x) .$$
2. Prove that $\mathrm{d}[M]_t \le ||A_t||_{\mathrm{OP}} \mathrm{Var}_{\nu_t}(\varphi)$.
3. Let $\varphi(x) = \mathbb{1}_E(x)$. Prove that
$$\mathrm{dVar}_{\nu_t}(\varphi) \le -\mathrm{d}[M]_t + \text{martingale}.$$
4. Conclude
$$\frac{\mathbb{E}[\mathrm{Var}_{\nu_t}(\varphi)]}{\mathrm{Var}_{\nu}(\varphi)} \ge \left(\mathbb{E}\left[\exp\left(\int_0^t ||A_s||_{\mathrm{OP}} \mathrm{d}s \right) \right] \right)^{-1} .$$

Exercise 4

For simplicity, suppose $C_t = \mathrm{Id}$.

1. Show that
$$\mathrm{d}a_t = A_t \mathrm{d}B_t . \tag{5}$$
2. Show that
$$\mathrm{d}A_t = \left(\int_{\mathbb{R}^n} (x - a_t)^{\otimes 3} \nu_t(x) \mathrm{d}x \right) \mathrm{d}B_t - A_t^2 \mathrm{d}t.$$
Hint: if you want computations to be easier, derive dA_t in dimension 1.

Exercise 5

Consider the case where μ is the standard Gaussian measure, namely
$$\frac{\mathrm{d}\mu}{\mathrm{d}x} = \frac{1}{(2\pi)^{\frac{n}{2}}} \exp\left(-\frac{||x||_2^2}{2} \right) .$$

1. Apply (4) to derive a formula for $\frac{\mathrm{d}\nu_t}{\mathrm{d}x}$, which depends on Z_t and θ_t.
2. Use (5) to compute A_t and a_t.
3. Write down the complete formula for $\frac{\mathrm{d}\nu_t}{\mathrm{d}x}$ with explicit a_t, Z_t.

Stochastic localization on discrete hypercube

Let $\mathcal{C}_n := \{-1,1\}^n$ be the n-dimensional hypercube. Let B_t be a standard Brownian motion on $\mathbb{R}^n$ adapted to a filtration $\mathcal{F}_t$. We consider the system of SDE

$$\begin{cases} \mathrm{d}F_t(x) = F_t(x)\langle C_t(x-a_t), \mathrm{d}B_t\rangle & \forall x \in \mathcal{C}_n\,, \\ F_0(x) = 1 \end{cases} \tag{6}$$

where C_t is an adapted process w.r.t B_t and

$$a_t = \int_{\mathcal{C}_n} x\nu_t(x)\mathrm{d}x\,.$$

Properties (A), (B) and (C) hold also on $\mathcal{C}_n$ [3].

Exercise 6

What is the dimension of (6) if $C_t = \mathrm{Id}$? Prove

(H.A) Almost surely, for all t, ν_t is a probability measure.

(H.B) For all $A \subset \mathcal{C}_n$, $\nu_t(A)$ is a martingale.

(H.C) The process a_t converges to a point $a_\infty := \lim_{t\to\infty} a_t \in \mathbb{R}^n$, which is distributed according to the law μ. Moreover, the measure ν_t almost surely weakly converges to a Dirac measure at a_∞.

We now prove a result from [2], which says that the Dirichlet form associated to ν_t is a supermartingale.

Exercise 7

We write $x \sim y$ to denote the adjacency relation on $\mathcal{C}_n$, i.e. x and y differ in exactly one coordinate. Let $\varphi : C_n \to \mathbb{R}$ be a test function.

1. Consider an arbitrary measure ν, let X and Y be two independent samples from ν.
   (a) Use the fact that $\mathrm{Var}(X) = \frac{1}{2}\mathbb{E}[(X-Y)^2]$ in order to show
   $$\mathcal{E}_\nu(\varphi,\varphi) = \frac{1}{2}\mathbb{E}_\nu \sum_{i=1}^n \mathbb{E}_\nu\left[\varphi(Y) = \varphi(X))^2 \middle| Y_{\sim i} = X_{\sim i}, X_{\sim i}\right].$$
   (b) Conclude
   $$\mathcal{E}_\nu(\varphi,\varphi) = \sum_{x\sim y} \frac{\nu(x)\nu(y)}{\nu(x)+\nu(y)}(\varphi(x)-\varphi(y))^2.$$
2. It follows that we need to show that
   $$\frac{\nu_t(x)\nu_t(y)}{\nu_t(x)+\nu_t(y)}$$
   is a supermartingale for fixed $x \sim y$.
   (a) Compute $\mathrm{d}\log\nu_t(x)$ and $\mathrm{d}\log\nu_t(y)$.
   (b) Compute $\mathrm{d}\log\big(\nu_t(x)+\nu_t(y)\big)$.
   (c) Apply Itô lemma in order to compute $\frac{\nu_t(x)\nu_t(y)}{\nu_t(x)+\nu_t(y)}$.

Exercise 8
Let $M_t = \int_{\mathcal{C}_n} \varphi \mathrm{d}\nu_t$. Prove

$$\mathrm{Var}_{\nu_0}(\varphi) = \mathbb{E}[M]_t + \mathbb{E}\mathrm{Var}_{\nu_t}(\varphi).$$

In particular

$$\mathrm{Var}_{\nu_0}(\varphi) \leq \mathbb{E}\mathrm{Var}_{\nu_t}(\varphi). \tag{7}$$

Exercise 9
Suppose you know that for any test function $\varphi : \mathcal{C}_n \to \mathbb{R}$

$$(1-\alpha)\mathrm{Var}_{\nu_t}(\varphi(X)) \leq \mathcal{E}_{\nu_t}(\varphi, \varphi)\,.$$

for some deterministic $0 < \alpha < 1$, i.e. you have a Poincaré inequality for the stochastic localized measure. Derive a Poincaré inequality for the measure ν_0.

Extra: Skorkohod embedding theorem

Skorkohod embedding theorem states the following.

Theorem 1. *Let X be a real-valued random variable with $\mathbb{E}[X] = 0$ and $\mathrm{Var}(X) < \infty$. Let W be a canonical real-valued Wiener process. Then there is a stopping time τ w.r.t. natural filtration such that*

$$W_\tau \overset{(d)}{=} X, \quad \mathbb{E}[\tau] = \mathbb{E}[X^2],$$

We prove $W_\tau \overset{(d)}{=} X$ using stochastic localization in dimension 1 (see [1] for more details).

Let μ be a measure satisfying

$$\int_{\mathbb{R}} x\mu(\mathrm{d}x) = 0 \quad \text{and} \int_{\mathbb{R}} x^2 \mu(\mathrm{d}x) < \infty.$$

For $c \in \mathbb{R}$ and $b \geq 0$ we write

$$V_\mu(b,c) = \int_{\mathbb{R}} \exp\left(cx - \frac{1}{2}bx^2\right)\mu(\mathrm{d}x)$$

and define the function

$$a_\mu(b,c) = V_\mu^{-1}(b,c)\int_{\mathbb{R}} x \exp\left(cx - \frac{1}{2}bx^2\right)\mu(\mathrm{d}x)\,,$$
$$A_\mu(b,c) = V_\mu^{-1}(b,c)\int_{\mathbb{R}} (x - a_\mu(b,c))^2 \exp\left(cx - \frac{1}{2}bx^2\right)\mu(\mathrm{d}x)\,.$$

Exercise 10
Determine for which values of b, c such that $a_\mu(b,c) = 0$ and $A_\mu(b,c) = \int_{\mathbb{R}} x^2 \mu(\mathrm{d}x)$.

Exercise 11
For stochastic localization, as before let a_t be the mean of the measure μ_t. Write the differential $\mathrm{d}a_t$ in terms of an arbitrary adapted driving process C_t. Use the form of the differential to choose C_t such that a_t is a Brownian motion up to a stopping time to be determined. Write down explicitly the system of SDE's.
Hint Define $T := T_\mu = \sup\{t \geq 0 : A_\mu(b_t, c_t) > 0\}$.

Exercise 12
Prove $\{T \leq t\}$ is measureable with respect $\mathcal{F}_t = \sigma(B_s; s \leq t)$.

Exercise 13
Let a_t be the mean of μ_t with the choice of C_t from above and let $\varphi : \mathbb{R} \to \mathbb{R}$ be a smooth test function.

1. Prove
$$\left|\int_{\mathbb{R}} \varphi(x)\mu_t(\mathrm{d}x) - \varphi(a_t)\right| \leq \sup_{x\in\mathbb{R}} |\varphi'(x)| A_t^{\frac{1}{2}} .$$

2. Prove
$$\lim_{t\to T^-} \int_{\mathbb{R}} \varphi(x)\mu_t(\mathrm{d}x) = \varphi(a_T) .$$

3. Apply Optional Stopping theorem to show
$$\mathbb{E}\left[\int_{\mathbb{R}} \varphi(x)\mu_{T\wedge t}(\mathrm{d}x)|\mathcal{F}_{s\wedge T}\right] \leq \sup_{x\in\mathbb{R}} |\varphi(x)| \quad \forall t \geq 0.$$

4. Apply Dominated Convergence theorem to show
$$\mathbb{E}\left[\int_{\mathbb{R}} \varphi(x)\mu_{T\wedge t}(\mathrm{d}x)|\mathcal{F}_{s\wedge T}\right] = \mathbb{E}\left[\int_{\mathbb{R}} \varphi(x)\mu_T(\mathrm{d}x)\middle|\mathcal{F}_{s\wedge T}\right].$$

5. Conclude $a_{T_\mu} \sim \mu$.

We now show that $\mathbb{E}[T_\mu] = \mathrm{Var}(\mu)$. For all $t > 0$ define $X_t = B_t - B_{t\wedge T}$. By previous exercises we know that for φ compactly supported test function
$$\mathbb{E}\left[\varphi(X_s)\middle|\mathcal{F}_{s\wedge T}\right] = \int_{\mathbb{R}} \varphi(x - B_{s\wedge T})\mu_{s\wedge T}(\mathrm{d}x) \quad \forall s \geq 0.$$

Exercise 14

Let $\varphi_n(x)$ be a monotone increasing sequence of positive, compact supported functions with $\lim_{n\to\infty}(x) = x^2$ for all $x \in \mathbb{R}$.

1. Prove $\mathrm{Var}(X_s|\mathcal{F}_{s\wedge T}) = A_{s\wedge T}$ for all $s \geq 0$,
2. Show $\mathrm{Cov}(X_s, B_{s\wedge T}) = 0$.
3. Prove $\lim_{t\to\infty} \mathbb{E}[B_{T\wedge t}^2] \leq \mathrm{Var}(\mu)$.
4. Prove $\mathbb{E}[T] \leq \mathrm{Var}(\mu)$.
5. Apply Optional Stopping theorem on $B_t^2 - t$ and conclude.

References

[1] Ronen Eldan. "Skorokhod embeddings via stochastic flows on the space of Gaussian measures". In: *Annales de l'Institut Henri Poincaré, Probabilités et Statistiques* 52.3 (2016), pp. 1259 – 1280.

[2] Ronen Eldan, Frederic Koehler, and Ofer Zeitouni. "A spectral condition for spectral gap: fast mixing in high-temperature Ising models". In: *Probability Theory and Related Fields* 182 (2022), pp. 1035–1051.

[3] Ronen Eldan and Omer Shamir. "Log concavity and concentration of Lipschitz functions on the Boolean hypercube". In: *Journal of Functional Analysis* 282.8 (2022).