Exercise Sheet 7

Recall Itô process: $dX_t = U_t\, dt + V_t\, dB_t$

Itô lemma: $g \in C^2$, $Y_t = g(X_t) \Rightarrow dY_t = \frac{\partial g}{\partial x}(X_t)\, dX_t + \frac{1}{2}\frac{\partial^2 g}{\partial x^2}(X_t)(dX_t)^2$

$$= \left(\frac{\partial g}{\partial x}(X_t) U_t + \frac{1}{2}\frac{\partial^2 g}{\partial x^2}(X_t) V_t^2 \right) dt + \frac{\partial g}{\partial x}(X_t) V_t\, dB_t$$

Exercise 1

Set $I_t = \int_{\mathbb{R}^n} \varphi(x)\, d\nu_t(x) = \int_{\mathbb{R}^n} \varphi(x) F_t(x)\, d\nu_0(x)$

Let $t < s$. Since $\nu_t(\cdot)$ is a martingale, we have

$$\mathbb{E}[I_t \mid \mathcal{F}_s] = \mathbb{E}\left[\int_{\mathbb{R}^n} \varphi(x) F_t(x)\, d\nu_0(x) \mid \mathcal{F}_s \right]$$

$$\overset{\text{Fubini}}{=} \int_{\mathbb{R}^n} \mathbb{E}[\varphi(x) F_t(x)\, d\nu_0(x) \mid \mathcal{F}_s]$$

$$= \int_{\mathbb{R}^n} \varphi(x) F_s(x)\, d\nu_0(x) = I_s.$$

Equation (2) follows directly from Sheet 1, Exercise 2

Exercise 2

$$\begin{cases} dF_t(x) = \langle x - a_t, dB_t \rangle \\ F_0(x) = 1 \end{cases}$$

Itô: $$d \log F_t(x) = \frac{dF_t(x)}{F_t(x)} - \frac{d(F_t(x))^2}{2F_t(x)^2}$$

$$= \langle x - a_t, dB_t \rangle - \frac{1}{2}\left(\langle x - a_t, dB_t \rangle\right)^2$$

$$= \langle x - a_t, dB_t \rangle - \frac{1}{2}\|x - a_t\|^2 dt$$

$$= \langle x - a_t, dB_t \rangle - \frac{1}{2}\|x\|^2 dt - \frac{1}{2}\|a_t\|^2 dt + \langle x, a_t \rangle dt$$

$$= \langle x, dB_t \rangle - \langle a_t, dB_t \rangle - \frac{1}{2}\|x\|^2 dt - \frac{1}{2}\|a_t\|^2 dt + \langle x, a_t \rangle dt$$

$$= \langle x, \underbrace{dB_t + a_t dt}_{= d\theta_t} \rangle - \frac{1}{2}\|x\|^2 dt + \underbrace{\left(- \langle a_t, dB_t \rangle - \frac{1}{2}\|a_t\|^2 dt\right)}_{\Rightarrow \text{ in normalization constant}}$$

$$\Rightarrow \quad F_t(x) = \frac{1}{Z_t} \exp\left\{ \langle x, \theta_t \rangle - \frac{t}{2}\|x\|^2 \right\}$$

Exercise 3

① $d M_t = d\nu_t(E) = d \int_E F_t(x)\, d\mu(x) = \int_E \langle x - a_t, dB_t \rangle\, d\mu(x)$

② $$d[M]_t = \left| \int_E \langle x - a_t, dB_t \rangle\, d\nu_t(x) \right|^2 dt$$
$$= \sup_{|\theta|=1} \left(\int_{\mathbb{R}^n} \varphi(x) \langle x - a_t, \theta \rangle\, d\nu_t(x) \right)^2$$
$$\leq \sup_{|\theta|=1} \int_{\mathbb{R}^n} \langle x - a_t, \theta \rangle^2\, d\nu_t(x)\ \mathrm{var}_{\nu_t}(\varphi)$$
$$= \| \mathrm{Cov}(\nu_t) \|_{op} \cdot \mathrm{var}_{\nu_t}(\varphi) = \| A_t \|_{op} \cdot \mathrm{var}_{\nu_t}(\varphi)$$

③ $$d\, \mathrm{var}_{\nu_t}(\varphi) = d\left(\mathbb{E}_{\nu_t}(\mathbb{1}_E^2) - \mathbb{E}_{\nu_t}(\mathbb{1}_E)^2 \right)$$
$$= d\,(M_t - M_t^2)$$
$$= dM_t - 2 M_t\, dM_t - \frac{1}{2} \cdot 2\, d[M]_t$$
$$= -\, d[M]_t + \text{martingale}$$

④ Combine ② and ③

$$d\, \mathrm{var}_{\nu_t}(\varphi) = -d[M]_t + \text{martingale}$$
$$\geq -\| A_t \|_{op} \cdot \mathrm{var}_{\nu_t}(\varphi)$$

By integration we get

$$\frac{\mathbb{E}[\mathrm{var}_{\nu_t}(\varphi)]}{\mathrm{var}_{\nu}(\varphi)} \geq \mathbb{E}\left[\exp\left\{ \int_0^t \| A_s \|_{op}\, ds \right]^{-1}$$

Exercise 4

(1) $$da_t = d\int x\,\nu_t(dx) = \int x\, d\nu_t(dx) = \int x(x-a_t)\,\nu_t(dx)\; dB_t$$
$$= A_t\, dB_t$$

(2)

$$dA_t = d\left(\int_{\mathbb{R}^n} x^{\otimes 2}\, F_t(x)\, d\mu(x) - a_t^{\otimes 2}\right)$$

$$= d\int x^{\otimes 2}\, d\nu_t(x) - d(a_t^{\otimes 2})$$
$$d(a_t^{\otimes 2}) = a_t \otimes da_t + da_t \otimes a_t + \frac{1}{2}\cdot 2\, d[a]_t$$
$$= a_t \otimes da_t + da_t \otimes a_t + A_t^2\, dt$$

$$dA_t = d\int (x-a_t)^{\otimes 2}\, \nu_t(dx) = \int d\left((x-a_t)^{\otimes 2}\right) \nu_t(dx)$$
$$= \boxed{-\int da_t \otimes (x-a_t)\,\nu_t(dx) - \int (x-a_t)\otimes da_t\, \nu_t(dx)}$$
$$+ \int (x-a_t)^{\otimes 2}\, d\nu_t(dx) - \boxed{2\int (x-a_t)\otimes d[a_t, \nu_t(dx)]_t}$$
$$+ \boxed{\int d[a_t, a_t]\,\nu_t(dx)} = -2A_t^2 + A_t^2 = -A_t^2$$

First two terms are 0 because

$$\int da_t \otimes (x-a_t)\,\nu_t(dx) = da_t \otimes \int (x-a_t)\,\nu_t(dx) = 0$$

Last two terms are of bounded variation

$$\Rightarrow \quad dA_t = \int (x-a_t)^{\otimes 2}\, d\nu_t(dx) - A_t^2\, dt$$
$$= \int (x-a_t)^{\otimes 3}\, \nu_t(dx)\, dB_t - A_t^2\, dt \qquad \square$$

Exercise 5

1 $$F_t(x) = \frac{1}{Z_t} \exp\left\{ \langle \theta_t, x\rangle - \frac{1}{2} t \|x\|^2 \right\}$$

$$\frac{d\nu_t}{dx} = \frac{1}{Z_t} \exp\left\{ \langle \theta_t, x\rangle - \frac{1}{2}(t+1)\|x\|^2 \right\}$$

2 The covariance matrix is $A_t = (t+1)^{-1}\,\mathrm{Id}$

From (5) we get $a_t = \int_0^t (s+1)^{-1}\, dB_s$

3 $$\frac{d\nu_t}{dx} = (t+1)^{n/2} (2\pi)^{-n/2} \exp\left\{ -\frac{1}{2}(t+1)\, \|(x-a_t)\|^2 \right\}$$

Exercise 6 See Proposition 9 of [3]

dim $= 2^n$

1 $$d\nu_t(\mathcal{C}_n) = d\int_{\mathcal{C}_n} F_t(x)\,\mu(x) = \int_{\mathcal{C}_n} (x-a_t)\, d\nu_t(x)\, dB_t = 0$$

2 Follows from def.

3 $$da_t = d\int x\,\nu_t(x) = \left(\int x \otimes (x-a_t)\,\nu_t(dx)\right) dB_t$$

$$= \left(\int (x-a_t)^{\otimes 2}\,\nu_t(dx)\right) dB_t = \mathrm{Cov}(\nu_t)\, dB_t$$

$\Rightarrow$ a_t is a martingale

$$d[\langle a_t, e_i\rangle]_t = \sum_{j=1}^n \mathrm{Cov}(\nu_t)_{i,j}^2 \geq \mathrm{Cov}(\nu_t)_{i,i}^2 = (1 - \langle a_t, e_i\rangle^2)^2\, dt$$

$\Rightarrow$ $\langle a_t, e_i\rangle$ converges to $\{\pm 1\}$ a.s

ν_t converges weakly to δ_{a_∞}

$\Rightarrow$ $\lim_{t\to\infty} a_t \in A \iff \lim_{t\to\infty} \nu_t(A) = 1 \quad \forall A \subset \mathcal{C}_n$

$\Rightarrow$ $\mathbb{P}(\lim_{t\to\infty} a_t \in A) = \lim_{t\to\infty} \mathbb{E}\,\nu_t(A) = \nu(A)$ $\square$

Exercise 7

$\varphi : \mathcal{C}_n \to \mathbb{R}$

① a) $$\mathcal{E}_\nu(\varphi,\varphi) = \mathbb{E}_\nu \sum_{i=1}^{n} \operatorname{var}\left(\varphi(X) \mid X_{\sim i}\right)$$

$$= \frac{1}{2}\mathbb{E}_\nu \sum_{i=1}^{n} \mathbb{E}_\nu\left[(\varphi(Y)-\varphi(X))^2 \mid Y_{\sim i} = X_{\sim i},\ X_{\sim i}\right]$$

b) $$= \frac{1}{2}\mathbb{E}_\nu \sum_{i=1}^{n} \frac{(\varphi(Y)-\varphi(X))^2 \cdot \mathbb{1}\{Y_{\sim i} = X_{\sim i}\}}{\mathbb{P}_\nu(Y_{\sim i} = X_{\sim i} \mid X_{\sim i})}$$

$$= \sum_{x\sim y} \nu(x)\nu(y)(\varphi(x)-\varphi(y))^2 \sum_{i=1}^{n} \frac{\mathbb{1}\{y_{\sim i} = x_{\sim i}\}}{\mathbb{P}_\nu(Y_{\sim i} = X_{\sim i})}$$

$$= \sum_{x\sim y} \frac{\nu(x)\nu(y)}{\nu(x)+\nu(y)}(\varphi(x)-\varphi(y))^2$$

② a) $$d\log \nu_t(x) = \langle dB_t, x-a_t\rangle - \frac{1}{2}\|x-a_t\|^2\,dt$$

$$d\log \nu_t(y) = \langle dB_t, y-a_t\rangle - \frac{1}{2}\|y-a_t\|^2\,dt$$

b) $$d\log(\nu_t(x)+\nu_t(y)) = \langle dB_t, \alpha(x-a_t)+\beta(y-a_t)\rangle - \frac{1}{2}\|\alpha(x-a_t)+\beta(y-a_t)\|^2\,dt$$

where $\alpha = \dfrac{\nu_t(x)}{\nu_t(x)+\nu_t(y)}$ $\qquad \beta = \dfrac{\nu_t(y)}{\nu_t(x)+\nu_t(y)}$

c) $$d\left(\frac{\nu_t(x)\nu_t(y)}{\nu_t(x)+\nu_t(y)}\right)$$

$$= \left(\frac{\nu_t(x)\nu_t(y)}{\nu_t(x)+\nu_t(y)}\right)\frac{1}{2}\left[\|\alpha(x-a_t)+\beta(y-a_t)\|^2 + \|\beta(x-a_t)+\alpha(y-a_t)\|^2 - \|x-a_t\|^2 - \|y-a_t\|^2\right]dt$$

$+$ martingale

Using the fact $de^{S_t} = e^{S_t}dS_t + \frac{1}{2}e^{S_t}d[S]_t$ and by convexity of $\|\cdot\|^2$

and $\alpha+\beta = 1$, $d\left(\dfrac{\nu_t(x)\nu_t(y)}{\nu_t(x)+\nu_t(y)}\right)$ is a supermartingale

Exercise 8

We know that M_t is a martingale and $M_\infty = \lim_{t\to\infty} M_t$ exists a.s. and has law $\varphi_* \nu$

$\Rightarrow$ $\operatorname{var}_\nu[\varphi \mid \mathcal{F}_t] = \operatorname{var}(M_\infty \mid \mathcal{F}_t)$ a.s. $\forall t$

Itô isometry: $\operatorname{var}_\nu(\varphi) = \mathbb{E}[M]_t + \mathbb{E}\operatorname{var}(M_\infty \mid \mathcal{F}_t)$

$$= \mathbb{E}[M]_t + \mathbb{E}\operatorname{var}_{\nu_t}(\varphi)$$

In particular, $\operatorname{var}_{\nu_0}(\varphi) \leq \mathbb{E}\operatorname{var}_{\nu_t}(\varphi)$

Exercise 9

$$(1-\alpha)\operatorname{var}_{\nu_0}(\varphi) \leq (1-\alpha)\,\mathbb{E}\operatorname{var}_{\nu_t}(\varphi)$$

$$\leq \mathbb{E}\mathcal{E}_{\nu_t}(\varphi,\varphi) \leq \mathcal{E}_{\nu_0}(\varphi,\varphi)$$

! this is true on $\mathcal{C}_n$

Exercise 10

Note that
$$a_\mu(0,0) = \int_{\mathbb{R}} x\,\mu(dx) = 0$$
$$A_\mu(0,0) = \int_{\mathbb{R}} x^2\mu(dx)$$

Exercise 11

The system of equations is

$$\begin{cases} dc_t = A_\mu^{-1}(b_t, c_t)\, dW_t + A_\mu^{-2}(b_t, c_t)\, a_\mu(b_t, c_t) dt & c_0 = 0 \\ db_t = A_\mu^{-2}(b_t, c_t)\, dt & b_0 = 0 \end{cases}$$

Let's understand why

$$da_t = d\int_{\mathbb{R}} x F_t(x)\,\mu(dx) = \int_{\mathbb{R}} x\, dF_t(x)\,\mu(dx)$$
$$= \left(\int_{\mathbb{R}} x(x-a_t)\, C_t\, F_t(x)\mu(dx)\right) dW_t$$

Note that

$$\int_{\mathbb{R}} a_t(x-a_t)\, C_t F_t(x)\,\mu(dx) = C_t\left(a_t\mu_t(\mathbb{R}) - \int_{\mathbb{R}} x\mu_t(dx)\right) = 0$$

Therefore

$$da_t = \left(\int_{\mathbb{R}} (x-a_t)^2 F_t(x)\mu(dx)\right) dW_t$$
$$= C_t\left(\int_{\mathbb{R}} (x-a_t)^2 F_t(x)\right) dW_t \overset{!}{=} dW_t$$

$\Rightarrow \quad C_t = A_t^{-1} \qquad \forall\, t \le T_\mu =: T$

Exercise 12

It follows by def of A

Exercise 13

$\forall\, 0 \le t \le T$

$$\left| \int_{\mathbb{R}} \varphi(x)\, \mu_t(dx) - \varphi(W_t) \right| = \left| \int_{\mathbb{R}} (\varphi(x) - \varphi(a_t))\mu_t(dx) \right|$$

$$\overset{C\text{-}S}{\le} \sup_{x\in\mathbb{R}} |\varphi'(x)| \int_{\mathbb{R}} (x - a_t)\mu_t(dx) \le$$

$$\le \sup_{x\in\mathbb{R}} |\varphi'(x)|\, A_t^{1/2}$$

Since $\lim_{t\to T^-} A_t = 0 \Rightarrow \lim_{t\to T^-} \int \varphi(x)\mu_t(dx) = \varphi(W_T)$

$\to$ μ_T is a Dirac probability measure with an atom at W_T

From Exercise 1 we know that $\int \varphi(x)\mu_t(dx)$ is a martingale

By OST

$$\mathbb{E}\left[\int_{\mathbb{R}} \varphi(x)\, \mu_{T\wedge t}(dx) \,\middle|\, \mathcal{F}_{s\wedge T} \right] = \int_{\mathbb{R}} \varphi(x)\, \mu_{T\wedge s}(dx) \qquad \forall t > s \ge 0$$

φ is bounded, $\mu_{T\wedge t}$ is a prob. measure:

$$\int_{\mathbb{R}} \varphi(x)\mu_{T\wedge t}(dx) \le \sup_{x\in\mathbb{R}} |\varphi(x)| \qquad \forall t \ge 0$$

DCT: $\lim_{t\to\infty} \mathbb{E}\left[\int_{\mathbb{R}} \varphi(x)\, \mu_{T\wedge t}(dx) \,\middle|\, \mathcal{F}_{s\wedge T} \right] = \mathbb{E}\left[\lim_{t\to T^-} \int_{\mathbb{R}} \varphi(x)\, \mu_t(dx) \,\middle|\, \mathcal{F}_{s\wedge T} \right]$

$$= \mathbb{E}\left[\int_{\mathbb{R}} \varphi(x)\, \mu_T(dx) \,\middle|\, \mathcal{F}_{s\wedge T} \right]$$

$$\Rightarrow \mathbb{E}\left[\int_{\mathbb{R}} \varphi(x)\, \mu_T(dx) \,\middle|\, \mathcal{F}_{s\wedge T} \right] = \int_{\mathbb{R}} \varphi(x)\, \mu_{s\wedge T}(dx)$$

Take $s=0$.

Exercise 14

OST: $\mathbb{E}[W_{T\wedge t}^2] = \mathbb{E}[T\wedge t] \quad \forall t>0$

Take $\lim_{t\to\infty}$ both sides, we want to show

$$\lim_{t\to\infty} \mathbb{E}[W_{T\wedge t}^2] = \mathbb{E}[W_T^2] - var(\mu)$$

Define $X_t = W_T - W_{t\wedge T}$

$$\mathbb{E}[\varphi(X_s) \mid \mathcal{F}_{s\wedge T}] = \int_{\mathbb{R}} \varphi(x - W_{s\wedge T})\, u_{s\wedge T}(dx)$$

Take $\varphi_n(x)$ as in the sheet

$$var(X_s \mid \mathcal{F}_{s\wedge T}) = \lim_{n\to\infty} \mathbb{E}[\varphi_n(X_s) \mid \mathcal{F}_{s\wedge T}]$$

$$= \lim_{n\to\infty} \int_{\mathbb{R}} \varphi_n(x - W_{s\wedge T})\, u_{s\wedge T}(dx) = A_{s\wedge T}$$

Since $\int_{\mathbb{R}} x\, u_{s\wedge T}(dx) = W_{s\wedge T} \Rightarrow Cov(X_s, W_{s\wedge T}) = 0$

$\Rightarrow var(X_s) + var(W_{s\wedge T}) = var(\mu)$

$\Rightarrow var(W_{t\wedge T}) \leq var(\mu) \quad \forall t \geq 0$

$\Rightarrow \lim_{t\to\infty} \mathbb{E}[W_{T\wedge t}^2] \leq var(\mu)$

$\Rightarrow \mathbb{E}[T] = \lim_{t\to\infty} \mathbb{E}[T\wedge t] \leq var(\mu)$

Apply OST $\mathbb{E}[W_T^2] = \mathbb{E}[T]$. $\square$