# Exercise Sheet 8

**We do not expect you to solve all the exercises in this sheet. You can use these exercises as useful references and perhaps to practice in the future.**

## Guided Exercise

**Exercise 1**

In this exercise we will prove the classical Rockafellar-Ruschendorf theorem. Let $c : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$

1. Define $\partial^c \varphi = \{(x, y) : \varphi(x) + \varphi^c(y) = c(x, y)\}$. Prove that if a set $G \subset \partial^c \varphi$ then $G$ is $c$-cyclically monotone.
2. Given a set $G \subset \mathbb{R}^n \times \mathbb{R}^n$ non-empty and $c$-cyclically monotone, prove that the following function is well defined (fix $(x_0, y_0)$:
$$\varphi(x) = \inf\{c(x, y_m) - c(x_0, y_0) + \sum_{i=1}^{m} c(x_i, y_{i-1}) - c(x_i, y_i)\}$$
when the infimum runs over choices of pair $(x_i, y_i) \in G$.
3. Show that $\varphi^{cc} = \varphi$ (you can use exercise 2 item 3).
4. Show that for any $(x, y) \in G$ , $z \in \mathbb{R}^n$,
$$c(x, y) - \varphi(x) \leq c(z, y) - \varphi(z)$$
(hint: take $t > \varphi(x)$.)
5. Conclude that $G \subset \partial^c \varphi$.
6. Conclude that for $c : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$, a set $G$ is $c$-cyclically monotone if and only if there exists some $\varphi$ in the $c$-class such that $G \subset \partial^c \varphi$.

## Additional Exercises

**Exercise 2**

Prove the following properties of the $c$-transform, when $c : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$:

1. Order reversing: $f \leq g$ implies $g^c \leq f^c$.
2. Involution on the image: $f^{ccc} = f^c$.
3. $f = f^{cc}$ if and only if $f(x)$ is an infimum of functions of the form $c(x, y) - \lambda$ where $\lambda \in \mathbb{R}, y \in \mathbb{R}^n$.

**Exercise 3**

In this exercise we will study a specific example of $c$-cyclic monotonicity, for $c(x, y) = \langle x, y \rangle$.

1. Write our the definition of $c$-cyclic monotonicity for this cost. This is called cyclic monotonicity.
2. Show that in the case of $n = 1$ a set is cyclically monotone if and only if it is a graph of a monotone increasing function (in the sense that $x_i \leq x_j$ implies $y_i \leq y_j$.)
3. Show that for $\varphi : \mathbb{R}^n \to \mathbb{R}$ differentiable and convex, that $\nabla \varphi$ is a cyclically monotone set.

4. Show that in higher dimensions for a family $(x_i, y_i)$ to be cyclically monotone it is not enough to check that it is monotone in pairs (in the sense that $x_i \leq x_j$ implies $y_i \leq y_j$.)

In the following exercises we will re-prove and use the Brenier-McCann theorem:

**Theorem 1** (Brenier-McCann)**.** *Let $\mu, \nu \in P(\mathbb{R}^n)$ and assume that $\mu$ is absolutely continuous with respect to the Lebesgue measure. Then, there exists a convex function $\varphi : \mathbb{R}^n \to \mathbb{R}^n$ such that $T = \nabla\varphi$ is defined $\mu$-almost everywhere, and $T_{\#}\mu = \nu$.*

**Exercise 4**

In this exercise, let $\mu$ be a measure on $\mathbb{R}^n$ which assigns no mass to any set of Hausdorff dimension (n- 1), and $\nu$ some measure with bounded support. We will prove the Brenier-McCann using a lovely geometric argument due to K. Ball.

1. Denote by $\nu_\alpha = \sum_{i=1}^n \alpha_i \delta_{y_i}$ for a finite set of points $y_i \in \mathbb{R}^n$ and $\sum \alpha_i = 1$. This is clearly a probability measure. Consider all functions of the form

$$\varphi_t(x) = \sup_i \langle x, y_i \rangle - 1/t_i$$

with $t = (t_i)_{i=1}^n$ and $\sum t_i = 1$. Prove using a rearrangement argument and Brouwer's fixed point theorem that there is a point $t$ in the unit simplex such that

$$\forall 1 \leq i \leq n \quad \mu(\{x : \varphi_t(x) = \langle x, y_i \rangle - 1/t_i\}) = \alpha_i.$$

2. Conclude that there exists a convex function $\varphi$ such that $T = \nabla\varphi$ exist $\mu$-almost everywhere and $T_{\#}\mu = \nu_\alpha$.

3. Approximate $\nu$ by discrete measures and conclude, using a convergence argument, the Brenier-McCann theorem.

4. ($\star$) Extend this argument to general costs and measures. What assumptions do you need on the cost?

**Exercise 5**

1. Write the statement of the Brenier-McCann Theorem for the measures $1/\mathrm{Vol}(K_1) Leb|_{K_1}$ and $1/\mathrm{Vol}(K_2) Leb|_{K_2}$ when $K_1, K_2$ two sets. In this case, what is $\mathrm{Vol}((I+T)(K_1))$?

2. Bound $\mathrm{Vol}((I+T)(K_1))$ from below by $1 + (\mathrm{Vol}(K_1)/\mathrm{Vol}(K_2))^{1/n}$ using the eigenvalue of the matrix $DT$.

3. Bound $\mathrm{Vol}((I+T)(K_1))$ from above and obtain the Brunn-Minkowski inequality.

**Exercise 6**

In this exercise we prove Talagrand's cost-entropy inequality, which states

$$C(f, g) \leq 2Ent(f||\gamma),$$

where $f$ is some density function $g$ is the Gaussian density, so that

$$C(f, g) = \int \|x - Tx\|^2 d\gamma, \quad Ent(f||\gamma) = \int f(x) \log(f(x)/g(x)) dx.$$

1. Write the statement of the Brenier-McCann Theorem, mapping $\gamma$ the Gaussian probability measure to the measure given by the density $f(x)$.

2. Rewrite $Ent(f||\gamma)$ in terms of $g$ and $T(x)$, where $T$ is the map transporting $\gamma$ to the probability with density $f$. You should end up with

$$Ent(f||\gamma) = \int g(x)\log(g(x)/(g(T(x))|DT(x)|).$$

3. Prove that our desired inequality reduces to showing

$$\int g(x)\log(|DT(x)|) \leq \int g(x)\langle x, T(x) - x\rangle.$$

4. Using the eigenvalue of the matrix $DT$, integration by parts, and a 1-dimensional inequality on log, prove the former inequality.