Exercise Sheet 2
Log-Sobolev inequality and Poincaré inequality

We do not expect you to solve all the exercises in this sheet. You can use these exercises as useful references and perhaps to practice in the future.

Guided Exercise - Poincaré Inequality

In this section, we will give some time in class to prove the simple exercises below, after which we will recap them on the board. Our goal is to prove the one-dimensional Gaussian Poincaré Inequality (also known as Nash's Poincaré Inequality), which states that

$$Var(f) \leq \mathbb{E}(|f'|^2)$$

whenever $f, f' \in L^2(\gamma)$.

1. Denote by $\gamma(x) = \mathrm{e}^{\frac{-|x|^2}{2}}$ Prove that for any $k \geq 0$ there exists a polynomial H_k of degree k such that

   $$\gamma^{(k)}(x) = (-1)^k H_k(x)\gamma(x).$$

   These are called Hermite polynomials.

2. Prove the following properties for all $x \in \mathbb{R}$ and $k \geq 0$:
   (a) $H_{k+1}(x) = xH_k(x) - H_k'(x)$,
   (b) $H_{k+1}'(x) = (k+1)H_k(x)$,
   (c) $H_k(-x) = (-1)^k H_k(x)$.

3. Deduce that the coefficient of x^k in H_k is 1.

4. Prove that

   $$\left\{\frac{H_k}{\sqrt{k}}\right\}_{k=0}^{\infty}$$

   is an orthonormal basis in $L^2(\gamma)$. Hint: Use integration by parts, (a) and (b).

5. Decompose $f \in L^2(\gamma)$ in the orthonormal basis you found.

6. Prove the 1-dimensional Gaussian Poincaré Inequality.

Additional Exercises

Exercise 1
Let $f : \mathbb{R}^n \to \mathbb{R}$ be convex and L-Lipschitz, i.e., $|f(x) - f(y)| \leq L|x - y|$ for all $x, y \in [a, b]^n$, then $Var[f(x_1, \ldots, x_n)] \leq L^2(b-a)^2$.

Exercise 2
Lef $f \in \mathcal{L}_1$, ν a measure on $\mathbb{R}$ with density $e^{-|x|}$. Show that $Var_\nu(f) \leq 4\mathbb{E}(|f'|^2)$.

Exercise 3
Let (X, d) be a metric space, $\alpha : \mathbb{R}_+ \to \mathbb{R}_+$ be a function such that for every 1-Lipschitz function $f : (X, d) \to \mathbb{R}$ and every $t > 0$

$$\mu\left(\left\{x : f(x) \geq \int f d\mu + t\right\}\right) \leq \alpha(t).$$

Then, show that for every Borel $A \subset X$ with $\mu(A) > 0$ and for every $t > 0$,

$$1 - \mu(A_t) \leq \alpha(\mu(A)t)$$

where A_t is the t-neighborhood of A.

Exercise 4

Using Prékopa Leindler, show that if a measure μ on $\mathbb{R}^n$ has a density f which is continuous on its support then f is a log-concave function if and only if μ is a log-concave measure, namely it satisfies

$$\mu((1-\lambda)A + \lambda B) \geq (1-\lambda)\mu(A) + \lambda\mu(B).$$

Exercise 5

Tensorize Log-Sobolev. Let $\{(X_i, \mu_i, d_i)\}_{i=1}^m$ be metric probability spaces. Assume for each i the log-Sobolev inequality is satisfied with constant β_i. Define for f on the product space $\prod_{i=1}^m X_i$,

$$|\nabla f|^2 = \sum_{i=1}^m |\nabla_i f_i|^2.$$

Show that on the product space, the log-Sobolev inequality is satisfied with constant $\max_i \beta_i$.

Exercise 6

Poincaré inequality under transformations. Let X be a random vector in $\mathbb{R}^d$ and suppose that $C_p(X) < \infty$.

- Let A be an $m \times n$ matrix and consider the random vector AX in $\gtrdot$. Prove that,

  $$C_p(AX) \leq \|A\|_{\text{op}}^2 C_p(X).$$

  Here, $\|A\|_{\text{op}}$ stands for the operator norm of A.

- More generally, if $f : \mathbb{R}^d \to \mathbb{R}^m$ is an L-Lipschitz function, prove that,

  $$C_p(AX) \leq L^2 C_p(X).$$

Exercise 7

In this exercise we extend the Gaussian Poincaré inequality we proved to $L_2(\gamma_n)$.

1. For any $k \in \mathbb{Z}_+^n$ and $x \in \mathbb{R}^n$ define

   $$\mathcal{H}_k(x) = \prod_{j=1}^n H_{k_j}(x_j).$$

   Prove that these polynomials are an orthonormal basis to $L^2(\gamma_n)$.

2. Denote by ∇f the usual gradient operator. Compute the adjoint operator, i.e. find an operator acting on $L^2(\gamma_n)$ such that

   $$\mathbb{E}[\nabla f g] = \mathbb{E}[f A g].$$

3. Prove that

   $$Var(f) \leq \mathbb{E}(|\nabla f|^2)$$

   whenever $f, \nabla f \in L^2(\gamma)$.

Exercise 8

Subadditivity of the Poincaré inequality. Let $\{\mu_i\}_{i=1}^n$ be probability measures on $\mathbb{R}^d$ and let $X_i \sim \mu_i$ be independent random vectors. Further let $(a_1, \ldots, a_d) = a \in \mathbb{R}^d$ be a fixed vector. Suppose that $C_p(\mu_i) < \infty$ for every $i \in [n]$ and let $S_a := \sum_{i=1}^n a_i X_i$. Use the previous two exercises to prove

$$C_p(S_a) \leq \sum_{i=1}^n a_i^2 C_p(\mu_i).$$

In particular, if $\mu_1 = \mu_2 = \cdots = \mu_n = \mu$, then

$$C_p(S_a) \leq \|a\|^2 C_p(\mu).$$

Exercise 9

Let $X \sim \mu$ stand for the Laplace distribution on $\mathbb{R}$. That is, $d\mu(x) = \frac{1}{2}e^{-|x|}dx$.

1. Prove the following integration by parts formula. For any test function f,

$$\mathbb{E}[f(X)] = f(0) + \mathbb{E}[\text{sign}(X)f'(X)].$$

2. Apply the integration by parts formula to squares of functions to deduce,

$$C_p(\mu) \leq 4.$$

3. By finding an appropriate sequence of functions prove that $C_p(\mu) = 4$.