Exercise Sheet 6
Review of differential geometry

Reading recommendations

- Manfredo P. do Carmo *Differential Geometry of Curves and Surfaces*
- John M. Lee, *Introduction to Smooth Manifolds*
- John M. Lee, *Riemannian Manifolds: An Introduction to Curvature*
- John W. Milnor, *Topology from the Differentiable Viewpoint*

Review exercises

Exercise 1

Prove that the following sets are smooth manifolds (e.g. by finding a set of charts):

1. The n-spheres $S^n = \{x \in \mathbb{R}^{n+1} : \sum_{i=1}^{n+1} x_i^2 \leq 1\}$.
2. Projective space.
3. The set of all k-dimensional linear subspaces of an n-dimensional space V (also known as the Grassmanian Manifold).

Exercise 2

Find an explicit Riemannian metric on the following spaces:

1. The unit 2-sphere S^2.
2. The unit torus $S^1 \times S^1$.
3. A body of revolution in $\mathbb{R}^3$, i.e. for a smooth curve $(\alpha(t), \beta(t))_{t\in(0,1)} \in \mathbb{R}^n$, the manifold given by $(\alpha(t)\cos(\theta), \beta(t)\cos(\theta), \sin(\theta))_{t\in(0,1),\theta\in[0,2\pi]}$.

Exercise 3

Suppose M is a smooth manifold, $p \in M$, and $X \in T_pM$. Prove:

1. If f is a constant function, then $Xf = 0$.
2. If $f(p) = g(p) = 0$, then $X(fg) = 0$.

Exercise 4

Find an explicit volume form for the following manifolds:

1. (Warmup) The Euclidean space $\mathbb{R}^n$.
2. The circle S^1.
3. The n-torus $S^1 \times \cdots \times S^1$.
4. The n-sphere S^n.

Exercise 5

Show that the geodesics on $\mathbb{R}^n$ with respect to the Euclidean connection are exactly the straight lines with constant speed parametrizations.

Exercise 6

Consider the curve $\gamma : [0,1] \to \mathbb{R}^2$ defined by $\gamma(t) = \left(t, t \sin \frac{1}{t}\right)$ for $t \in (0,1]$ and $\gamma(0) = (0,0)$.

1. Show that γ is continuous on $[0,1]$, and is smooth on $(0,1]$.
2. Show that γ has infinite length.

Exercise 7

Compute the curvature of the following curves:

1. The curve given by $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ and the parametrization
$$\gamma(t) = (a \cosh t, b \sinh t), \quad t \in \mathbb{R}$$
for some $a, b > 0$.
2. The ellipse
$$\gamma(t) = (a \cos t, b \sin t), \quad t \in \mathbb{R}$$
for some $a, b > 0$.