Lecture 1 : motivation

Euclidean QFT: ↙ goal

probability measures μ on $S'(\mathbb{R}^d)$

Stochastic quantization: ↙ tool

Gaussian $W \xrightarrow{F_\mu} \phi$

$\text{Law } \phi = \mu$

(e.g. ϕ solves some stoch diff equ with W)

- physical system : {states} {observables}
- Classical mech : Eg : (q, p) functions like $E = q^2 + p^2$.
- Quantum mechanics :

  Hilbert space $H :=$ { states A }

  Operators = { observables w }

  "measurement": $w(A) = \langle w, Aw \rangle$

- Dynamics :

  Time translation $\overbrace{\text{group}}^{\text{unitary}}$ $U(t)$

  $U(t) = e^{-itE}$ where E: Hamiltonian/energy

  $w(t) = U(t) w$ "Schrödinger picture"

  $A(t) = U(t)^{-1} A \, U(t)$ "Heisenberg picture" relation: $\omega(t)(A) = \omega(A(t))$ Exercise

$\omega(t)(A) = \langle \omega(t), A\,\omega(t)\rangle$
$= \langle U(t)\omega, A\, U(t)\omega\rangle$
$= \langle \omega, A(t)\,\omega\rangle$

$$\boxed{\text{Key quantities: } \omega(A_1(t_1)\cdots A_n(t_n))}$$

Assume: $\exists\, \varphi \in \mathcal{H}$ (vacuum, ground state)
$U(t)\varphi = \varphi$

$\varphi(A_1(t_1)\cdots A_n(t_n))$
$= \langle \varphi, U(t_1)^{-1} A_1 U(t_1) \cdots U(t_n)^{-1} A_n U(t_n)\, \varphi\rangle$

$$\boxed{= \langle \varphi, A_1 U(t_1 - t_2) A_2 \cdots U(t_{n-1} - t_n) A_n \varphi\rangle}$$

Example: $\mathcal{H} = L^2(\mathbb{R}, \rho)$ $\rho \sim N(0, \frac{1}{2a})$

operators: e.g. multiplication by $C(\mathbb{R})$

To describe a dynamic, recall:

Hermite polynomials $H_n(x)$ is orthogonal basis of H.

check: normalized?

$U(t) H_n = e^{-i a n t} H_n$

Ground state $\varphi = H_0 = 1$

one can also describe this dynamic as follows:

$$U(t) = e^{-itE}$$ E is energy operator

where E is an operator with is diagonalized by H_n

$$E H_n = a_n H_n$$

TA session: E is OU operator. $-\partial_x^2 + x\partial_x$ (up to constants)

"Euclidean" = "going to imaginary time"

replace t by $-it$. $e^{-itE} \to e^{-tE}$ ($t \geq 0$)

"Wick rotation"

"analytic continuation"

" $K(t)$

Continuous semigroup of contractions

$$\boxed{\begin{array}{c} \langle \varphi, A_1 K(t_1 - t_2) A_2 \cdots, K(t_{n-1} - t_n) A_n \varphi \rangle \\ \text{"Schwinger functions"} \end{array}}$$

(1) Can they recover quantum system?

(2) They are correlation functions of a $\boxed{\text{prob measure.}}$
(of a random function)

Ex:

$$dX_t = -\alpha X_t\,dt + dB_t$$
$$X_0 \sim N(0, \tfrac{1}{2\alpha}) \qquad t \in \mathbb{R}$$

Prob measure μ on $C(\mathbb{R})$

$$X_t = \int_{-\infty}^{t} e^{-\alpha(t-s)}\,dB_s$$

random field on $\mathbb{R}$

$$E(X_t X_{t'}) = E\left[\int_{-\infty}^{t} e^{-\alpha(t-s)}\,dB_s \int_{-\infty}^{t'} e^{-\alpha(t'-s)}\,dB_s\right]$$

$t \geq t'$

$$= e^{-\alpha(t-t')} \int_{-\infty}^{t'} e^{-2\alpha(t'-s)}\,ds = \frac{e^{-\alpha(t-t')}}{2\alpha}$$

$$E[f(X_t)\,|X_0 = x] = E\left[f\left(e^{-\alpha t}x + N\left(0, \frac{1-e^{-2\alpha t}}{2\alpha}\right)\right)\right]$$
$$=: (K(t)f)(x) \qquad \text{Semigroup}$$

Compute:

$$K(t)\,e^{i\lambda x}$$
$$= E\left[e^{i\lambda\left(e^{-\alpha t}x + N\left(0, \frac{1-e^{-2\alpha t}}{2\alpha}\right)\right)}\right]$$
$$= e^{i\lambda e^{-\alpha t}x} \cdot e^{-\frac{\lambda^2}{2}\left(\frac{1-e^{-2\alpha t}}{2\alpha}\right)}$$

Thus

$$K(t)\underbrace{e^{i\lambda x + \frac{\lambda^2}{4\alpha}}}_{\sum_{n\geq 0}\frac{(i\lambda)^n}{n!}H_n(x)} = \underbrace{e^{i\tilde{\lambda}x + \frac{\tilde{\lambda}^2}{4\alpha}}}_{\sum_{n\geq 0}\frac{(i\lambda)^n}{n!}e^{-\alpha n t}H_n} \qquad (\tilde{\lambda} = \lambda e^{-\alpha t})$$

$$\Rightarrow \quad \underbrace{K(t) H_n = e^{-\alpha n t} H_n}$$

This is exactly $U(t)$ in imaginary time!

More interestingly, correlations are just Schwinger functions!

For $A_1, A_2 \cdots A_n \in C(\mathbb{R})$ (observables)

$$\mathbb{E}[A_1(X_{t_1}) \cdots A_n(X_{t_n})]$$

$t_1 \; t_2 \cdots t_{n-1} \; t_n$

$$= \mathbb{E}[A_1(X_{t_1}) \cdots \mathbb{E}[A_n(X_{t_n}) \mid X_{t_{n-1}}]]$$

$$= \mathbb{E}[A_1(X_{t_1}) \cdots A_{n-1}(X_{t_{n-1}})\, K(t_n - t_{n-1}) A_n (X_{t_{n-1}})]$$

$$= \cdots$$

$$= \mathbb{E}[(A_1 K(t_2 - t_1) A_2 \cdots K(t_n - t_{n-1}) A_n)(X_{t_1})]$$

$$= \langle \varphi, A_1 K(t_2 - t_1) \cdots K(t_n - t_{n-1}) A_n \varphi \rangle_{L^2(\mathbb{R}, \rho)}$$

$\varphi = 1$

Stationary $\rho = N(0, \frac{1}{2\alpha})$

In general, Schwinger functions must satisfy some properties to recover quantum system.

e.g.: $\langle \varphi, A_2 K(t) A_1 A_1 K(t) A_2 \varphi \rangle \geq 0$

"reflection positivity" (RP)

Osterwalder-Schrader theorem (baby version)

RP + certain regularity property $\xRightarrow{\text{recover}}$ quantum data (H, $U(t)$ etc)

RP: $\mathbb{E}[\overline{\Theta F(X)}\, F(X)] \geq 0$ $\forall$ functionals F

$\uparrow$ flip X: $X(t) \to X(-t)$ $\qquad F(X) = \sum_k c_k e^{i\lambda_k X(t_k)}$

remark: above OU field X is RP.

Beyond Gaussian:

$$\mathbb{Q}_T(dw) = \frac{1}{Z_T} e^{-\int_{-T}^{T} V(X_s(w))\, ds} \underset{\swarrow \text{OU}}{\mathbb{P}(dw)}$$

reflection positive:

$$G = e^{-\int_0^T V(X_s)\, ds}$$

$$\Theta G = e^{-\int_0^T V(X_{-s})\, ds}$$

$$\boxed{\mathbb{E}_{\mathbb{Q}_T}(\overline{\Theta F}\, F) \overset{?}{\geq} 0}$$

$$e^{-\int_{-T}^{T} V(X_s)\, ds} = \Theta G \cdot G = \overline{\Theta G} \cdot G$$

$$\mathbb{E}_{\mathbb{Q}_T}[\overline{\Theta F}\, F] = \frac{1}{Z_T} \mathbb{E}_{\mathbb{P}}[\overline{\Theta F}\, F \underbrace{e^{-\int_{-T}^{T} V(X_s(w))\, ds}}_{\overline{\Theta G} \cdot G}]$$

$$= \frac{1}{Z_T} \mathbb{E}_{\mathbb{P}}[\overline{\Theta(FG)}\, FG] \underset{\text{RP of } \mathbb{P}}{\geq} 0$$